A STOCHASTIC PROCESS STUDY OF TWO-ECHELON SUPPLY CHAIN WITH BULKY DEMAND PROCESS INCORPORATING COST SHARING COORDINATION STRATEGIES

By

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This research considers a single-item two-echelon supply chain facing a sequence of stochastic bulky customer demand with random order inter-arrival time and random demand size. The demand process is a general renewal process and the cost functions for both parties involve the renewal function and its integral. The complexity of the general renewal function causes the computational intractability in deciding the optimal order quantities, so approximations for the renewal function and its integral are introduced to address the computational complexity. Asymptotic expansions are commonly used in the literature to approximate the renewal function and its integral when the optimal decisions are relatively large compared to the mean of the inter-renewal time. However, the optimal policies do not necessarily fall in the asymptotic region. So the use of asymptotic expansions to approximate the renewal function and its integral in the cost functions may cause significant errors in decision making. To overcome the inaccuracy of the asymptotic approximation, this research proposes a modified approximation. The proposed approximation provides closed form functions for the renewal function and its integral which could be applied to various optimization problems such as inventory planning, supply chain management, reliability and maintenance. The proposed approximations are tested with commonly used distributions and applied to an application in the literature, yielding good performance. By applying the proposed approximation method to the supply chain cost functions, this research obtains the optimal policies for the decentralized and the centralized cases. The numerical results provide insights into the cost savings realized by the centralization of the supply chain compared to the decentralized case. Furthermore, this research investigates coordination schemes for the decentralized case to improve the utilities of parties. A cost sharing mechanism in which the vendor offers the retailer a contract as a compensation of implementing vendor-desired inventory policy is investigated. The sharing could be realized by bearing part of the retailer's inventory holding cost or fixed cost. The contract is designed to minimize the vendors cost while satisfying the individual rationality of the retailer. Other forms of coordination mechanisms, such as the side payment and delayed payment, are also discussed.

Key words: Stochastic Processes, Supply Chain Coordination, Renewal Function

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DEDICATION

To my parents, brother, and sister.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

AN IMPROVED APPROXIMATION FOR THE RENEWAL FUNCTION AND ITS INTEGRAL

1.1 Introduction and literature review

Let the random variable X_n denote the intercurrence or inter-renewal time between the $(n-1)^{\text{th}}$ and n^{th} events in a renewal process. Assume $X_1, X_2, \ldots$ to be a sequence of non-negative, independent random variables having a common probability distribution $F(x) = P\{X_k \leq x\}, x \geq 0, k = 1,2, \ldots$,and $\mu_1 = E(X_i), 0 < \mu_1 < \infty$.

Define $S_0 = 0, S_n = \sum_{i=1}^{n} X_i\,, n = 1,2, \ldots$, so that S_n would be the time epoch at which the n^{th} event occurs. For each $t \geq 0$, $N(t)$ is the largest integer $n \geq 0$ so that $S_n \leq t$. The random variable of $N(t)$ represents the number of events up to time t and the renewal function $M(t)$ is defined by $M(t) = E[N(t)], \quad t \geq 0$ (Tijms 2003).

Define the cumulative distribution function (cdf) $F_n(t) = P\{S_n \leq t\}, t \geq 0, n = 1,2, \ldots$, and $F_1(t) = F(t)$. It is implied that $P\{N(t) \geq n\} = F_n(t), n = 1,2, \ldots$. Given an $F(t)$, the renewal function $M(t)$ satisfies the integral equation (1.1)

$$M(t) = F(t) + \int_0^t M(t-x)dF(x) \tag{1.1}$$

The integral equation has a unique solution of $M(t)$, which is bounded on finite intervals under the assumption that $F(t)$ is continuous in t, $F(0) = 0$, and $F(\infty) = 1$ (Cox 1962). Let $f(t)$ denote the corresponding probability density function (pdf), if exists, for $F(t)$. Then,

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx. \tag{1.2}$$

The renewal function $M(t)$ and its integral $I(t) = \int_0^t M(x)dx$ play an important role in decision makings involving the renewal process, such as inventory planning, supply chain planning, reliability and maintenance analysis (e.g., Bahrami et al. 2000, Barlow and Proschan 1965, Sheikh and Younas 1985, Tijms 1994). However, obtaining the renewal function, $M(t)$, analytically is complicated and even impossible for most distribution functions. As an analytical method, the Laplace transform $M(s)$ of the renewal function satisfies

$$M(s) = \frac{f_0(s)}{s\left(1 - f_0(s)\right)}, \tag{1.3}$$

where $f_0(s)$ is the Laplace transforms of the density function of the inter-renewal time, $f(t)$ (From 2001). It is usually difficult to obtain $M(t)$ through the inversion of $M(s)$ (Jaquette 1972). We can obtain an exact computation of the renewal function, $M(t)$ for all $t \geq 0$ analytically only for a few special cases of $F(t)$ (Tijms 2003), such as the exponential distribution. Furthermore, in many real-life applications the distribution for the inter-renewal time may not be known. Therefore, approximations of the renewal function have drawn much interest in the literature and result in various methods.

The asymptotic expansion is very helpful in the approximation of the renewal function and its integral because of its simplicity (Tijms 2003). The asymptotic approximation only requires the first several moments (the first two moments for the renewal function approximation and the first three for its integral) and does not need the exact distribution function for the inter-renewal times. Because it provides a closed-form, the asymptotic approximation has been widely applied to the optimization problems that involve the renewal process, such as inventory planning, reliability and maintenance planning (e.g., Cetinkaya et al. 2008). However, asymptotic expansions for $M(t)$ are not accurate for small values of t and may yield poor optimal solutions. This research tries to address this drawback of asymptotic approximations by proposing a new approach of approximation. At the same time, our proposed approximations keep the positive features of asymptotic approximations such as simplicity, closed-form expression for optimization, and independence from the distributions of inter-renewal times.

It is rather easy to compute $M(t)$ numerically for a given value of t (Jaquette 1972) and a variety of approaches have been developed in the literature, such as cubic-splining algorithm by McConalogue (McConalogue 1981) to compute the renewal function by numerical convolution, the generating function algorithm by Giblin (1984), and power series expansion. The power series method is used for Weibull distribution in most studies (e.g., Weiss 1981, White 1964) but can be extended to all distributions with a power series expansion (Smeitink and Dekker 1990). Smith and Leadbetter (1963) found an iterative solution for the case in which the inter-renewal time follows a Weibull distribution. Another iterative solution method with the Weibull distributed inter-renewal time was given by White (1964). A numerical integration approach, which covers Weibull, Gamma, Lognormal, truncated Normal and inverse Normal distributions, was offered by Baxter et al. (1982). Garg and Kalagnanam (1998) proposed a Pade approximation (a class of rational polynomial approximants (Baker and Graves-Morris 1996)) approach to solve the renewal equation for the inverse Normal distribution. Their method uses Pade approximants to compute the renewal function near the origin and switches to the asymptotic values farther from the origin. They presented a polynomial switchover function in terms of the coefficient of variation of the distribution, enabling one to determine *a priori* if the asymptotic value can be used instead of computing the Pade approximant. A shortcoming of their method is that it does not provide a compact closed-form until applying the numerical method of Xie (1989). Kaminskiy and Krivtsov (1997) used a Monte Carlo simulation, which provides a universal numerical solution to the renewal function equation, covering essentially any parametric or empirical distribution used to model time-to-failure distributions. A method called the RS-method was established by Xie (1989) for solving renewal-type integral equations based on direct numerical Riemann-Stieltjes integration. The RS-method is particularly useful when the probability density function has singularities. The numerical method of Xie (1989) was used as a starting point of a numerical approximation proposed by From (2001) that constructs a two-piece modified rational function with the second piece being a linear function of t. An approximation for the renewal function of a failure distribution with an increasing failure rate was proposed by Jiang (2010). Although all the methods that numerically compute the renewal function are generally accurate for the small values of t but do not provide a closed-form expression that is useful for decision makings. Our proposed approximation not only is accurate in the range of small values of t but also provides a closed-form expression to facilitate optimization.

Another approach to compute the renewal function is the approximation based on $F(t)$ of a given distribution by the well known equation (Cox 1962):

$$M(t) = \sum_{0}^{\infty} F_n(t), \tag{1.4}$$

where $F_n(t)$ is the cdf of S_n, the epoch of the n^{th} renewal and is the convolution of $f(t)$ and $F_{n-1}(t)$. For most of the distributions, it is difficult to calculate $F_n(t)$. Therefore, Gamma distribution, whose $F_n(t)$ are easy to obtain, is often used to approximate

$F_2(t), F_3(t), \ldots$ based on the first two moments. The idea of exact computation of the first few terms of the renewal function in the series and approximation of the other terms using a two-moment match was developed by Smeitink and Dekker (1990). Their suggested approximation for $M(t)$ is in the form of $F(t) + \sum_{n=2}^{\infty} \overline{F_n}(t)\,, t > 0$, where $\overline{F_n}(t)$ is the distribution function of $\bar{X}_1 + \cdots + \bar{X}_n$ and $\bar{X}_1, \ldots, \bar{X}_n$ are independent and have a common gamma(α, λ) distribution. The values of α and λ parameters are determined such that the first two moments of the original inter-renewal times X_i are matched by the first two moments of the gamma(α, λ) distribution (Tijms 2003). Their numerical experiments show that their approximation yields quick and useful approximation of the renewal function provided that coefficient of variation is not too large. However, Gamma's distribution function is already too complicated for optimization in addition to possibly complicated $F(t)$. Our proposed approximation is independent of the inter-renewal time distribution and easy to apply for decision making.

A very important issue that has been only discussed in a small number of studies (e.g., Baxter et al. 1982) is the computation of the renewal function integral. The integral of a renewal function is extensively used in the studies that deal with the waiting and/or accumulating counting process such as inventory holding cost in inventory planning problems or cumulative damage process in reliability and maintenance problems (Zacks 2010). Baxter et al. (1982) developed a recursively defined algorithm to numerically compute the values of renewal function and its integral for a given t but their method is not useful for cases where a closed-form expression is required for optimizing an objective function. Our work provides closed-form approximations for both the renewal function and its integral.

Approximations of the renewal function are required to meet the following three requirements of simplicity, accuracy and applicability (Jiang 2010). Simplicity requires that the approximation has a closed-form expression and can be directly used without the need of further numerical computation. The approximation should be accurate enough from an engineering perspective within the potential value range of decision variables. The applicability means that the range of t in which the approximation is accurate should be large and it is applicable for a wide range of distribution families rather than a specific distribution. Some researchers have tried to address all of these requirements (e.g., Giblin 1984, Kaminskiy and Krivtsov 1997, Spearman 1989), but they could not meet all of them, missing either one or more of the requirements due to the complicated nature of the renewal function. In this paper, we propose a simple, yet accurate and applicable, approximation of the renewal functions and their integrals. The numerical results show that our approximation performs well in the entire range of t and is easy enough to get the closed-form expressions with respect to t and plug into objective functions (e.g. cost functions) to be optimized.

The remainder of the chapter is organized as follows. Section 1.2 discusses the asymptotic approximation of the renewal function and its integral. Section 1.3 presents the proposed approximation called Modified Approximation followed by Section 1.4 where numerical results are presented to verify the proposed Modified Approximation. An application is shown with an example from the literature to demonstrate the outperformance of the proposed approximation versus the commonly used asymptotic approximation in Section 1.5. Finally, Section 1.6 concludes the chapter.

1.2 Asymptotic approximation of the renewal function

The asymptotic expansion is useful in the approximation of the renewal function and its integral because of its simplicity. As a result of the elementary renewal theorem and key renewal theorem the following limits hold as asymptotic approximations of the renewal function and its integral (Tijms 2003):

$$\lim_{t\to\infty}\left[M(t)-\frac{t}{\mu_1}\right]=\frac{\mu_2}{2\mu_1^2}-1, \text{and} \tag{1.5}$$

$$\lim_{t\to\infty}\left[I(t)-\left\{\frac{t^2}{2\mu_1}+\left(\frac{\mu_2}{2\mu_1^2}-1\right)t\right\}\right]=\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}, \tag{1.6}$$

where μ_1, μ_2 and μ_3 denote the first, second and third moments of the inter-renewal time distribution. Being a result of asymptotic behavior of the renewal function, Equations (1.5) and (1.6) approximate the renewal function when t approaches infinity or is large enough. In real-life applications, however, t in the solution space may not be large enough to justify the usage of the asymptotic approximation. Some studies (e.g., Cetinkaya et al. 2008) use the asymptotic expansions (1.5) and (1.6) regardless of the value of t, which may result in errors in the calculation of the renewal function and therefore the optimal decisions. The numerical experiments in Tijms (1994) show that whether the asymptotic approximation is appropriate is related to the squared coefficient of variation of the inter-renewal time, c_X^2, which equals $\frac{\mu_2-\mu_1^2}{\mu_1^2}$. When $c_X^2=1$ (i.e., exponentially distributed inter-renewal times), the asymptotic expansion of (1.5) and (1.6) are exact. Numerical experiments in Tijms (1994) show that the asymptotic approximation works accurately enough when $t \geq t_0$ where,

$$t_0 = \begin{cases} \dfrac{3}{2} c_X^2 \mu_1 & \text{if } c_X^2 > 1, \\ \mu_1 & \text{if} 0.2 < c_X^2 \leq 1, \\ \dfrac{1}{2c_X} \mu_1 & \text{if } 0 < c_X^2 \leq 0.2. \end{cases} \tag{1.7}$$

Equation (1.7) presents a general guidance about when the asymptotic approximation is appropriate only based on the first two moments of the inter-renewal time distribution. Please note that the asymptotic approximation does not work very well when c_X^2 is much larger than 1 or close to 0. Both cases yield a large value of t_0. Equation (1.7) also indicates that the threshold value heavily depends on μ_1. When $t < \mu_1$, the asymptotic approximation does not work well. As pointed by Tijms (1994), the asymptotic expansion especially deteriorates as $c_X^2 \to 0$. For a distribution with a given variance, t_0 increases rapidly in μ_1 because both $\frac{1}{2c_X}$ and μ_1 in the third case of (1.7) increase. Figures 1.1 through 1.3 show several typical examples of the renewal function and its asymptotic approximation for different distributions and different c_X^2 values. The comparison of the approximated renewal function and the corresponding simulation results shows that there is a big gap between the actual value of renewal function and its estimation from the asymptotic approximation when t is small. Our approach is to build a function with a smaller gap versus simulation result than that of asymptotic approximation for the small values of t. The asymptotic approximation yields negative values when t is small in Figures 1.1 and 1.2, which can favor the decision of zero when some cost terms are positively correlated to the renewal function.

Knowing more information about the inter-renewal time distribution in addition to the first two moments could make the computation of the renewal function easier and/or more accurate (Gou et al. 2008, Heisig 1998, Jin and Liao 2009). Gou et al. (2008)

assume that the customer arrivals follow a Poisson process. Heisig (1998) presents an exact expression for $M(t)$ only for the case that demand is distributed according to a K_2-distribution. Jin and Liao (2009) proposed different approaches for different cases. They discussed that the closed form solution for the renewal function exists only for a few failure time distributions (e.g., exponential distribution), and many other distributions such as Weibull, Normal and Lognormal have to be solved numerically.

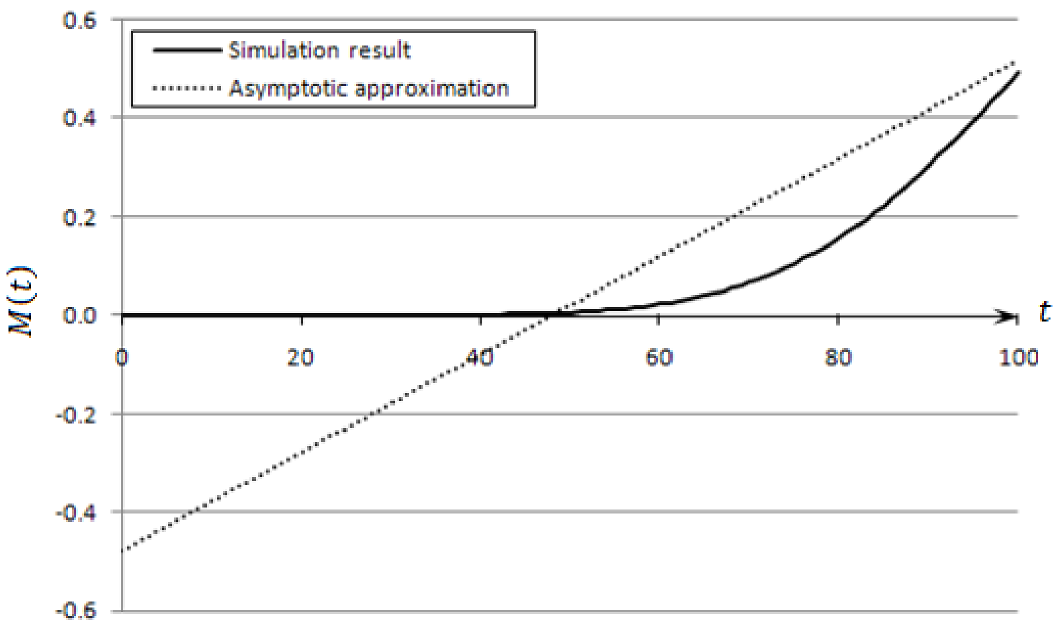

Figure 1.1 Simulated renewal function and asymptotic approximation - Normal distribution (100, 20) with $\mu_1 = 100$, $\sigma^2 = 400$ and $c_X^2 = 0.04$.

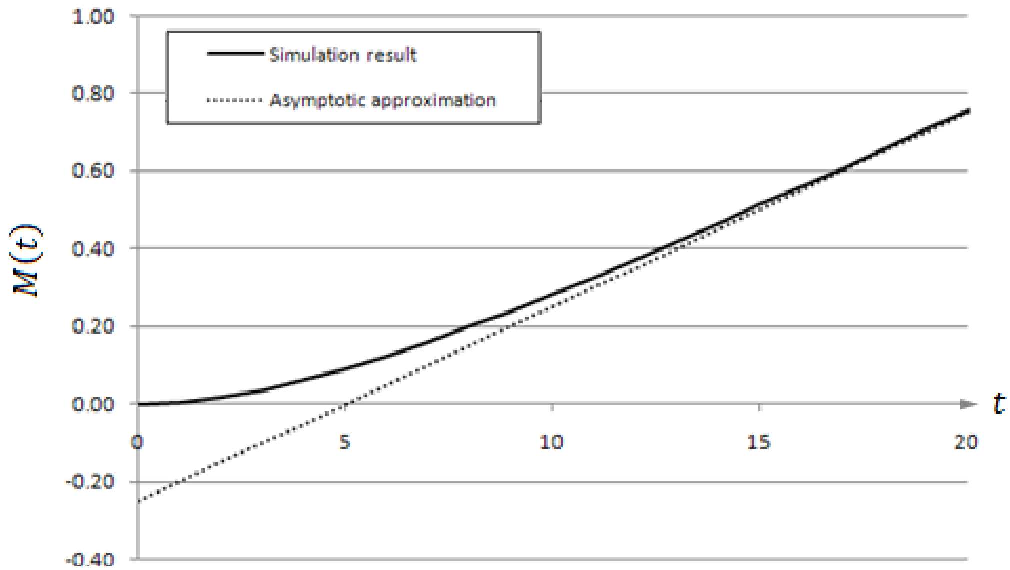

Figure 1.2 Simulated renewal function and asymptotic approximation - Erlang distribution (2, 10) with $\mu_1 = 20$, $\sigma^2 = 200$ and $c_X^2 = 0.5$.

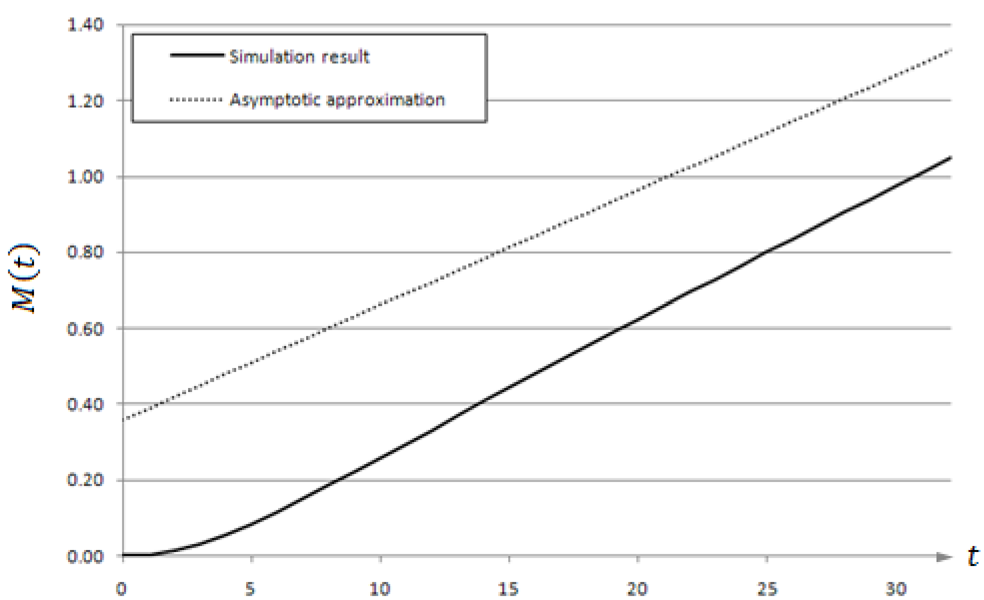

Figure 1.3 Simulated renewal function and asymptotic approximation - Lognormal distribution (3, 1) with $\mu_1 = 33.12$, $\sigma^2 = 1{,}884.32$ and $c_X^2 = 1.72$.

To take advantage of the simple approximation for standard Normal distribution with high accuracy by Zelen and Severo (1964), Wang and Pham (1999) approximate the renewal function for a special case of Normal distribution in their study of maintenance and reliability. To fill the gap in the current studies, especially, to answer the question of what the approximation should be when $t \leq t_0$, we will introduce our proposed approximation in the next section.

1.3 Modified approximation

The modified approximation is proposed for two cases under which the approximation is slightly different. In the first case the distribution of inter-renewal time is known while in the second case we do not know the distribution but know the first and second moments (or mean and variance) of inter-renewal time distribution. Subsections 1.3.1 and 1.3.2 discuss these two cases respectively.

1.3.1 Modified approximation knowing the inter-renewal time distribution

If we plot a typical renewal function (e.g., Figure 1.1), we find that the general shape of the renewal function starts from the $(0,0)$ point, remains at very small values for a while, and then approaches the asymptotic approximation line with a smooth transition. Inspired by this observation, we define a three-piece approximation, $\widetilde{M}(t)$, with two switch-over points t_1 and t_2 where $t_1 < t_2$.

$$\widetilde{M}(t) = \begin{cases} 0 & t < t_1, \\ \widehat{M}(t) & t_1 \leq t < t_2, \\ \dfrac{t}{\mu_1} + \dfrac{\mu_2}{2\mu_1^2} - 1 & t \geq t_2. \end{cases} \tag{1.8}$$

As shown in Figure 1.4 and, $\widehat{M}(t)$ denotes the modified approximation when t is between t_1 and t_2. Equation (1.7) can be used in determining the point of t_2 based on the value of c_X^2 and μ_1, but Equation (1.7) provides no clue about the value of t_1. If we know the distribution of the inter-renewal times, we can determine t_1 and t_2 points based on a predefined probability and the inverse function of the associate inter-renewal time distribution, $F(t)$, as

$$F(t_1) = P_1 \quad \text{and} \quad F(t_2) = P_2. \tag{1.9}$$

Our extensive numerical experiments show a good choice of t_1 and t_2 would be given by

$$P_1 = 0.02 \quad \text{and} \quad P_2 = \begin{cases} 0.5 & \text{if } c_X^2 < 1 \\ 0.9 & \text{if } c_X^2 > 1 \end{cases}. \tag{1.10}$$

To calculate $\widehat{M}(t)$, we define a linear function that goes through the points $(t_1, 0)$ and $(t_2, \frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1)$, where the asymptotic approximation is used at the point of $t = t_2$. Please see Figure 1.4 for an illustration. Therefore, the modified approximation function between $[t_1, t_2)$would be

$$\widehat{M}(t) = \frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}(t - t_1). \tag{1.11}$$

We use the same structure to have the approximation of $\tilde{I}(t)$ for the integral of the renewal function, $I(t)$,

$$\tilde{I}(t) = \begin{cases} 0 & t < t_1 \\ \hat{I}(t) & t_1 \le t < t_2 \\ \frac{t^2}{2\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right)t + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} & t \ge t_2. \end{cases} \tag{1.12}$$

Since $\widehat{M}(t)$ is a linear function, we define $\hat{I}(t)$ as a quadratic function that goes through the points $(t_1, 0)$ and $(t_2, \frac{t_2^2}{2\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right)t_2 + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2})$ and make $\tilde{I}(t)$ continuous at both t_1 and t_2. Please see Figure 1.5 for an illustration. Furthermore, the slope of $\tilde{I}(t)$ is set at zero when $t = t_1$ to make $\tilde{I}(t)$ smooth at $t = t_1$ because simulations show that the integral of the renewal function, $I(t)$, increases from zero with a small rate at the beginning. Therefore,

$$\hat{I}(t) = \left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + [\frac{\mu_2}{2\mu_1^2} - 1]t_2 + \frac{1}{2\mu_1}t_2^2}{(t_1 - t_2)^2} \right) t^2 + \left(\frac{-2t_1([\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}] + [\frac{\mu_2}{2\mu_1^2} - 1]t_2 + \frac{1}{2\mu_1}t_2^2)}{(t_1 - t_2)^2} \right) t + \left(\frac{[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}]t_1^2 + [\frac{\mu_2}{2\mu_1^2} - 1]t_1^2 t_2 + \frac{1}{2\mu_1}t_1^2 t_2^2}{(t_1 - t_2)^2} \right). \tag{1.13}$$

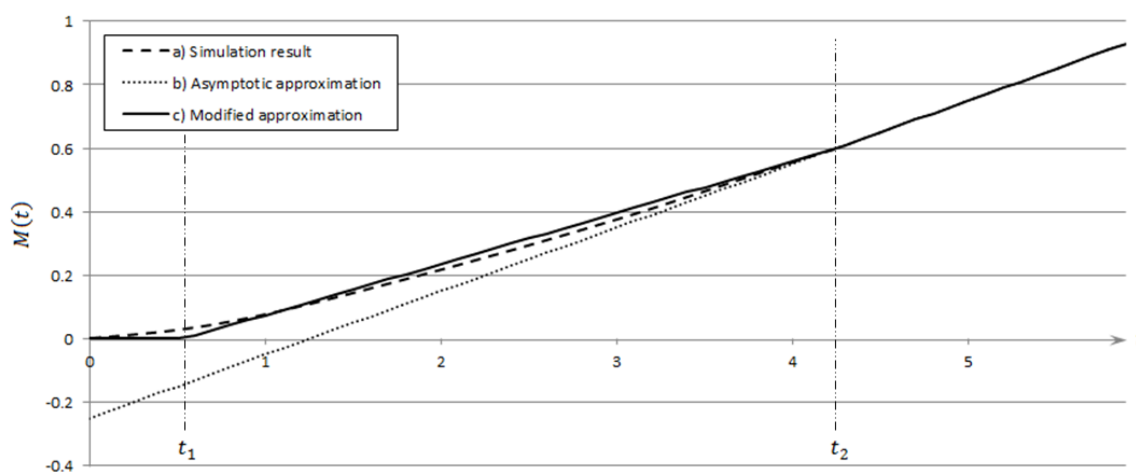

Figure 1.4 The simulated and approximated renewal function for Erlang distribution (2, 2.5)

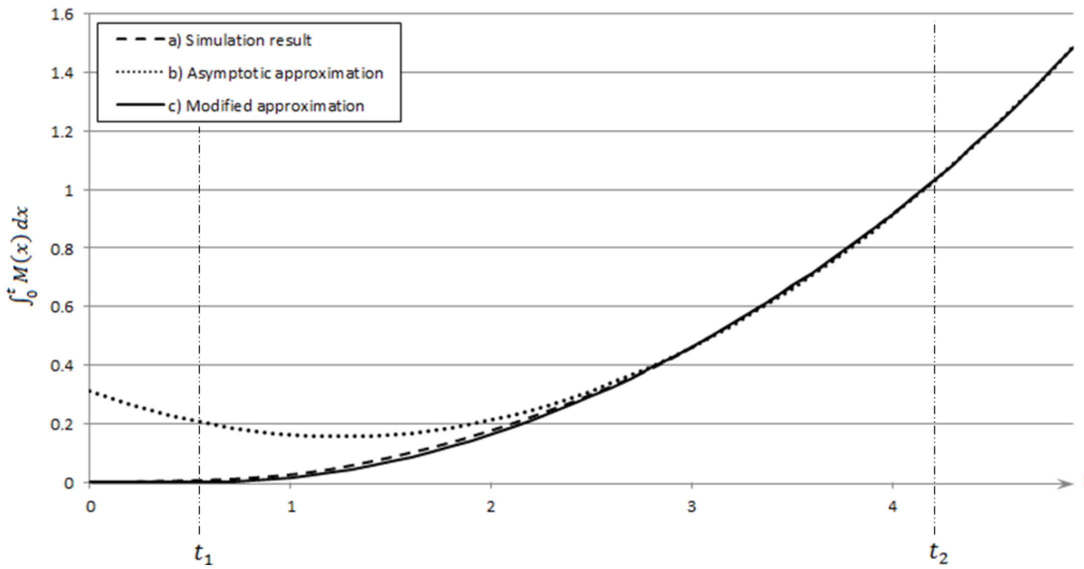

Figure 1.5 The simulated and approximated integral of the renewal function for Erlang distribution (2, 2.5)

1.3.2 Modified approximation not knowing the inter-renewal time distribution

In practice, the distribution for the inter-renewal time distribution could be unknown while the first several moments are available. In this case, the way of determining t_1 and t_2 points by Equation (1.9) needs to be changed in order to use the modified approximations defined by Equations (1.8) and (1.12). Let σ^2 denote the variance of the inter-renewal time. Then, the values of t_1 and t_2 are given by.

$$t_1 = \max\,(\mu_1 - k_1\sigma, 0) \;\text{ and }\; t_2 = \mu_1 + k_2\sigma. \tag{1.14}$$

Our numerical experiments show that the coefficients of $k_1 = 1$ and $k_2 = 0$ define good values of t_1 and t_2 respectively.

Numerical experiments show that the determination of t_2 based on the distribution rather than the first two moments as in Equation (1.7) yields better approximations. Table 1.1 shows a comparison of average deviation of approximated renewal function from simulated values for the two approaches of obtaining t_2, based on the distribution or based on the first two moments as in Equation (1.7).

Table 1.1 Comparison of average deviation of approximated $M(t)$ and its integral for the two cases of obtaining t_2 from simulated renewal function

Distribution	c_X^2	Average deviation from the simulated $M(t)$		Average deviation from the simulated $\int_0^t M(x)\,dx$	
		$t_2 = F^{-1}(P_2)$	t_2 based on Equation (1.3)	$t_2 = F^{-1}(P_2)$	t_2 based on Equation (1.3)
Erlang (2, 2.5)	0.5	0.0057	0.0088	0.0180	0.0203
Erlang (2, 10)	0.5	0.0065	0.0093	0.0590	0.0700
Lognormal (1, 0.5)	0.284	0.0349	0.0405	0.0790	0.0829
Lognormal (3, 1)	1.718	0.0461	0.0566	0.0400	0.0769
Weibull (1, 5)	0.052	0.2640	0.2589	0.1929	0.2127
Weibull (35, 345)	0.273	0.0147	0.0168	0.0845	0.0980

1.3.3 Modified approximation not knowing the inter-renewal time distribution

In practice, the distribution for the inter-renewal time distribution could be unknown while the first several moments are available. In this case, the way of determining t_1 and t_2 points by Equation (1.9) needs to be changed in order to use the

1.4 Numerical results

Numerical experiments are conducted to evaluate the effectiveness of the proposed modified approximation for the renewal function and its integral. Simulation results are used as a baseline in order to compare the asymptotic and the proposed approximations for various distributions with different parameters. Tables 1.2, 1.3 and 1.4 show sample numerical results for three different distributions with the coefficient of variation falling in different ranges as in Equation (1.7), $c_X^2 > 1$, $0.2 < c_X^2 \leq 1$, and

$0 < c_X^2 \leq 0.2$. Each table shows simulation results, asymptotic approximation and modified approximation of $M(t)$ and $I(t)$ for different t values.

Table 1.2 shows numerical results for Erlang distribution (2, 2.5) with $c_X^2 = 0.5$. To obtain the modified approximation value in this table, we have $t_1 = 0.54$ and $t_2 = 4.19$ respectively, following Equations (1.9) and (1.10). As the table shows, the deviations of modified approximation are much smaller than that of asymptotic approximation, especially for small values of t. Figures 1.4 and 1.5 illustrate the results in Table 1.2. We consider a Normal distribution with the mean of 100 and standard deviation of 20 in Table 1.3. The smaller deviations of modified approximation from the simulated values compared to that of asymptotic approximation show a good performance of modified approximation while c_X^2 is very small

Table 1.2 Results for Erlang distribution (2, 2.5) with $\mu_1 = 5.0$ and $c_X^2 = 0.5$

	Sim.[1] (1)	Asy. App.[2] (2)	Dev.* (2) vs.(1)	Mod. App.[3] (3)	Dev. (3) vs.(1)
t			$M(t)$		
0	0.00	-0.25	0.25	0.00	0.00
1	0.06	-0.05	0.11	0.07	0.01
2	0.20	0.15	0.05	0.24	0.04
3	0.37	0.35	0.02	0.40	0.02
4	0.56	0.55	0.01	0.56	0.00
t			$I(t) = \int_0^t M(x)\,dx$		
0	0.000	0.3125	0.31	0.000	0.00
1	0.022	0.1625	0.14	0.016	0.01
2	0.148	0.2125	0.06	0.164	0.02
3	0.432	0.4625	0.03	0.464	0.03
4	0.894	0.9125	0.02	0.917	0.02

*) Deviation is calculated as the absolute value of difference between simulation result and approximation.

1) Simulation,

2) Asymptotic approximation, and

3) Modified approximation.

The numerical results show that the modified approximation outperforms the asymptotic approximation for other distribution function of inter-renewal time, for example, Lognormal and Weibull distribution functions in Tables 1.4 and 1.5

Table 1.3 Results for Normal distribution (100, 20) whose $c_X^2 = 0.04$

	Sim. (1)	Asy. App. (2)	Dev. (2) vs.(1)	Mod. App. (3)	Dev. (3) vs.(1)
t			$M(t)$		
0	0.00	-0.48	0.48	0.00	0.00
10	0.00	-0.38	0.38	0.00	0.00
20	0.00	-0.28	0.28	0.00	0.00
30	0.00	-0.18	0.18	0.00	0.00
40	0.00	-0.08	0.08	0.00	0.00
50	0.01	0.02	0.01	0.00	0.01
60	0.02	0.12	0.10	0.01	0.01
70	0.07	0.22	0.15	0.14	0.07
80	0.16	0.32	0.16	0.27	0.11
90	0.31	0.42	0.11	0.39	0.09
t			$I(t) = \int_0^t M(x)\,dx$		
0	0.00	8.37	8.37	0.00	0.00
10	0.00	4.07	4.07	0.00	0.00
20	0.00	0.77	0.77	0.00	0.00
30	0.00	-1.53	1.53	0.00	0.00
40	0.01	-2.83	2.83	0.00	0.01
50	0.03	-3.13	3.16	0.00	0.03
60	0.11	-2.43	2.53	0.00	0.10
70	0.56	-0.73	1.29	0.52	0.05
80	1.74	1.97	0.23	1.97	0.23
90	3.92	5.67	1.75	5.67	1.75

Table 1.6 compares the deviation of asymptotic and modified approximation from simulation results for Weibull distributions with different parameters. The numerical results show that for different c_X^2 values, the modified approximation has less deviation

from simulation results on average compared to the asymptotic approximation. Less deviation are also observed for each of the Weibull distributions. Please see Table A.1 in Appendix A.

Table 1.4 Results of $M(t)$ for Lognormal distribution (3, 1) with $\mu_1 = 33.1$, $\sigma^2 = 1884.3$ and $c_X^2 = 1.72$

	Sim. (1)	Asy. App. (2)	Dev. (2) vs.(1)	Mod. App. (3)	Dev. (3) vs.(1)
t			$M(t)$		
0	0.00	0.36	0.36	0.00	0.00
1	0.00	0.39	0.39	0.04	0.04
2	0.01	0.42	0.41	0.08	0.07
3	0.03	0.45	0.42	0.11	0.09
4	0.05	0.48	0.43	0.15	0.10
5	0.08	0.51	0.43	0.19	0.11
6	0.12	0.54	0.42	0.23	0.11
7	0.15	0.57	0.42	0.27	0.12
8	0.18	0.60	0.42	0.30	0.12
9	0.22	0.63	0.41	0.34	0.12
10	0.26	0.66	0.40	0.38	0.12
t			$I(t) = \int_0^t M(x)\,dx$		
0	0.000	-49.684	49.68	0.000	0.00
1	0.000	-49.31	49.31	0.015	0.01
2	0.005	-48.905	48.91	0.060	0.06
3	0.024	-48.471	48.49	0.136	0.11
4	0.064	-48.006	48.07	0.242	0.18
5	0.131	-47.511	47.64	0.378	0.25
6	0.229	-46.986	47.21	0.544	0.32
7	0.358	-46.43	46.79	0.740	0.38
8	0.524	-45.845	46.37	0.967	0.44
9	0.725	-45.229	45.95	1.224	0.50
10	0.972	-44.583	45.55	1.511	0.54

Table 1.5 Results of $M(t)$ for Weibull distribution (1, 5) with $\mu_1 = 0.918$, $\sigma^2 = 0.044$ and $c_X^2 = 0.052$

t	Sim. (1)	Asy. App. (2)	Dev. (2) vs.(1)	Mod. App. (3)	Dev. (3) vs.(1)
			$M(t)$		
0	0.00	-0.36	0.36	0.00	0.00
1	0.00	-0.34	0.34	0.00	0.00
2	0.00	-0.31	0.31	0.00	0.00
3	0.01	-0.28	0.28	0.00	0.01
4	0.01	-0.25	0.26	0.00	0.01
5	0.02	-0.22	0.24	0.00	0.02
6	0.02	-0.19	0.22	0.00	0.02
7	0.03	-0.17	0.20	0.02	0.01
8	0.04	-0.14	0.18	0.04	0.00
9	0.05	-0.11	0.16	0.05	0.00
10	0.06	-0.08	0.14	0.07	0.00
t			$I(t) = \int_0^t M(x)\,dx$		
0	0.000	3.08318	3.08	0.000	0.00
1	0.000	2.7339	2.73	0.000	0.00
2	0.002	2.41284	2.41	0.000	0.00
3	0.006	2.11998	2.11	0.000	0.01
4	0.014	1.85533	1.84	0.000	0.01
5	0.027	1.6189	1.59	0.000	0.03
6	0.047	1.41067	1.36	0.001	0.05
7	0.073	1.23065	1.16	0.013	0.06
8	0.107	1.07884	0.97	0.040	0.07
9	0.150	0.95524	0.80	0.081	0.07
10	0.207	0.85985	0.65	0.138	0.07

Table 1.6 Deviation for Weibull distributions with different c_X^2

c_X^2	$M(t)$		$I(t) = \int_0^t M(x)\,dx$	
	Asy. App.	*Mod. App.*	*Asy. App.*	*Mod. App.*
0.004	0.159	0.101	0.555	0.409
0.053	0.030	0.018	0.421	0.399
0.273	0.027	0.016	0.729	0.662
0.461	0.035	0.026	1.098	1.050
5.000	0.342	0.272	16.814	14.565
Average	0.118	0.086	3.923	3.417

Table 1.7 Average deviation of approximated $M(t)$ from simulation results for three methods of determining switch-over points

	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	*Mod. App.*	*Asy. App.*	*Mod. App.*
Distribution	Method I			
Erlang (2,2.5)	0.0743	0.0190	0.0523	0.0211
Normal (100,20)	0.1736	0.0304	0.1251	0.0706
Lognormal (3,1)	0.3319	0.1470	0.3313	0.1503
Distribution	Method II			
Erlang (2,2.5)	0.0632	0.0454	0.0269	0.0490
Normal (100,20)	0.1736	0.0284	0.1085	0.0622
Lognormal (3,1)	0.3579	0.1777	0.3579	0.1831
Distribution	Method III			
Erlang (2,2.5)	0.0632	0.0264	0.0436	0.0292
Normal (100,20)	0.1039	0.0525	0.0720	0.0677
Lognormal (3,1)	0.2677	0.0874	0.2666	0.0884

Table 1.8 Average deviation of approximated $\int_0^t M(x)\,dx$ from simulation results for three methods of determining switch-over points

	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	*Mod. App.*	*Asy. App.*	*Mod. App.*
Distribution	Method I			
Erlang (2,2.5)	0.0951	0.0162	0.0679	0.0188
Normal (100,20)	2.3664	0.4675	1.5087	1.1175
Lognormal (3,1)	41.4273	10.3084	41.2438	10.5375
Distribution	Method II			
Erlang (2,2.5)	0.0816	0.0741	0.0372	0.0980
Normal (100,20)	2.3664	0.4049	1.6272	1.3251
Lognormal (3,1)	43.3314	12.2720	43.1389	12.6439
Distribution	Method III			
Erlang (2,2.5)	0.0816	0.0240	0.0575	0.0272
Normal (100,20)	1.6190	1.5434	1.2122	2.0054
Lognormal (3,1)	36.3162	8.2881	36.1589	8.3856

The three methods are I) determining t_1 and t_2 based on $F(t_1) = P_1$ and $F(t_2) = P_2$ in the case of knowing the inter-renewal time distribution, II) determining $t_1 = \mu_1 - k_1\sigma$ and $t_2 = \mu_1 + k_2\sigma$ in the case of not knowing the inter-renewal time distribution, and III) Determining t_2 based on the value of coefficient of variation as in Equation (1.7). Since method III does not offer any formula to determine t_1, method I is used instead. The same comparison is also made for the Weibull distributions with different parameters in Appendix A in Tables A.2 through A.7.

1.5 An application example

The proposed approximation method can be used in many application areas, such as maintenance and reliability, inventory planning, supply chain planning, in which the challenge is to derive an explicit expression for a renewal function that is often used in an optimization model. As an example to show the performance of our proposed method, we discuss a Vendor-Managed Inventory (VMI) problem studied by Cetinkaya et al. (2008).

Consider a single item inventory system involving one retailer and one vender with stochastic demand. For some reason, the retailer does not keep any stock. Each customer order arriving at the retailer is immediately transmitted to the vendor. The vendor must satisfy all orders eventually, though he could consolidate multiple orders into one outbound shipment to the retailer to achieve economies of scale inherent in transportation. Once the inventory level of the vendor drops below zero, a replenishment order is placed to bring the inventory back to its base stock. The replenishment lead time is assumed to be zero and the time between two consecutive replenishments is defined as a replenishment cycle. In this setting, successive orders from customers reach the retailer following a stochastic process with inter-arrival times T_n. The stochastic process $\{T_n, n = 1,2, \ldots\}$ consists of independently and identically distributed nonnegative random variables with finite mean λ. The orders are considered to be bulky where the size of the n^{th} order from customers to the retailer is denoted by Y_n. The stochastic process $\{Y_n, n = 1,2, \ldots\}$ also consists of independently and identically distributed nonnegative random variables with distribution $G(\cdot)$ with finite mean μ_1. The inventory decision variables of interest are the order-up-to level Q_R and the critical dispatch quantity Q_D, which is a threshold value that triggers a dispatch to the retailer. Letting $S_n = \sum_{i=1}^{n} T_i \;, n \geq 1$ and $S_0 = 0$, we define $N_1(t) = \sup\{n: S_n \leq t\}$. S_n is the arrival time of the n^{th} retailer order and $N_1(t)$ denotes the renewal process that registers the number of retailer orders placed by time t. Letting $D_n = \sum_{i=1}^{n} Y_i \;, n \geq 1$ and $D_0 = 0$, we also define $N_2(y) = \sup\{n: D_n \leq y\}$. D_n is the cumulative demand immediately after the n^{th} retailer order and $N_2(y)$ denotes the renewal process that counts the number of retailer orders consolidated up to y units. It is assumed that the two stochastic processes of $\{T_n\}$ and $\{Y_n\}$ are independent. The costs of the system include A_D, fixed cost of dispatching, A_R, fixed cost of replenishing the vendor inventory, h, unit inventory holding cost at the vendor's warehouse, and w, waiting penalty cost per unit per unit time at the retailer side.

Under the above assumptions, the inventory process is a regenerative process and the regeneration points are the vendor's replenishment moments, when the target inventory level Q_R is reached. Based on the renewal-reward theorem (Tijms 2003), the long-run average cost rate is

$$C(Q_D, Q_R) = \frac{E(Replenishment\ cycle\ cost)}{E(replenishment\ cycle\ lenth)}. \tag{1.15}$$

The optimal policy parameters Q_D and Q_R are then computed by solving the optimization problem of

$$\min C(Q_D, Q_R)$$
$$\text{s.t. } Q_D \geq 0, Q_R \geq 0. \tag{1.16}$$

Let U_i denote the length of the i^{th} consolidation cycle (that is the time between two consecutive dispatches) and W_i denote the size of the consolidated load accumulated during the i^{th} consolidation cycle. Therefore, $U_i \sim S_{N_2(Q_D)+1} = \sum_{n=1}^{N_2(Q_D)+1} T_n$ and $W_i \sim D_{N_2(Q_D)+1} = \sum_{n=1}^{N_2(Q_D)+1} Y_n$ and consequently $E[U_i] = E\left[S_{N_2(Q_D)+1}\right] = \frac{1}{\lambda} E[N_2(Q_D) + 1] = \frac{1}{\lambda}(M_G(Q_D) + 1)$ and $E[W_i] = E\left[D_{N_2(Q_D)+1}\right] = \mu_1 E[N_2(Q_D) + 1] = \mu_1(M_G(Q_D) + 1)$ where $M_G(\cdot)$ is the renewal function for $G(\cdot)$ and characterizes the renewal process of $\{Y_n, n = 1,2, \ldots\}$. Let K denote the number of consolidation cycles within a given replenishment cycle so that $E(\text{replenishment cycle lenth}) = E(\sum_{i=1}^{K} U_i)$. Letting $M_H(.)$ denote the renewal function of $H(\cdot)$, the distribution function of W_i, we have $E(\sum_{i=1}^{K} U_i) = \lambda E(M_G(Q_D) + 1)E[K] = \lambda(M_G(Q_D) + 1)(M_H(Q_R) + 1)$, and

$$E(\text{replenishment cycle lenth}) = \lambda(M_G(Q_D) + 1)(M_H(Q_R) + 1). \tag{1.17}$$

The expected cost in one replenishment cycle is the summation of the expected customer waiting cost, expected inventory holding cost, replenishment cost, and expected dispatching cost to the retailer.

$$E(\text{replenishment cycle cost}) = A_R + A_D(M_H(Q_R) + 1) + w\lambda \int_0^{Q_D} y dM_G(y)\,(M_H(Q_R) + 1) + h\lambda\left(Q_R + \int_{Q_D}^{Q_R} (Q_R - y) dM_H(y)\right) M_G(Q_D) + 1. \tag{1.18}$$

Combining the above results, we have (please see Appendix A.3 for how to obtain this result)

$$
\begin{aligned}
C(Q_D, Q_R) = {} & \frac{A_R + A_D(M_H(Q_R) + 1)}{\lambda(M_G(Q_D) + 1)(M_H(Q_R) + 1)} + \frac{w \int_0^{Q_D} y dM_G(y)}{M_G(Q_D) + 1} \\
& + \frac{h\left(Q_R + \int_{Q_D}^{Q_R} (Q_R - y) dM_H(y)\right)}{(M_H(Q_R) + 1)}.
\end{aligned}
\tag{1.19}
$$

Equation (1.19) cannot be optimized in its current shape because of the complexity of the renewal functions and their integrals. Approximations of the renewal function and its integral are necessary to obtain optimal values of Q_D and Q_R. We solve this problem using the proposed modified approximations and compared the results with those from the asymptotic approximation, presented by Cetinkaya et al. (2008).

It is noteworthy to mention that the retailer order to the vendor is essentially the summation of Q_D and excess life of the process defined by $N_2(y)$ beyond Q_D. Equations (1.8) and (1.12) for approximating $M_H(y)$ require the first, second and third moments of $H(\cdot)$, denoted by ϑ_1, ϑ_2 and ϑ_3 respectively. We have

$$\vartheta_1 = \mu_1(M_G(Q_D) + 1), \tag{1.20}$$

$$\vartheta_2 = \mu_2(M_G(Q_D) + 1) + 2\mu_1\left(Q_D M_G(Q_D) - \int_0^{Q_D} M_G(x)\, dx\right) \tag{1.21}$$

and

$$\vartheta_3 = \mu_3 + 3\mu_1 \int_0^{Q_D} x^2 m_G(x) dx + 3\mu_2 \int_0^{Q_D} x m_G(x) dx + \mu_3 \int_0^{Q_D} m_G(x)\, dx. \tag{1.22}$$

Here, $m_G(x)$ is the first-order derivative of $M_G(x)$. To be able to calculate ϑ_3, we need to calculate $\int_0^t x^2 m(x) dx$ or $\int_0^t x^2 dM(x)$ for a general renewal process as

$$\int_0^t x^2 m(x)dx = t^2 M(t) - 2\int_0^t x d\int_0^s M(x)\,dx$$
$$= t^2 M(t) - 2\left[t\int_0^t M(x)\,dx - \int_0^t \int_0^x M(s)\,dsdx\right]. \tag{1.23}$$

Theorem 1. Suppose that $F(x)$ is non-arithmetic with $\mu_2 = E(X^2) < \infty$ then,

$$\lim_{t\to\infty}\left[\int_0^t \int_0^s M(x)dx\,ds - \left\{\left[\frac{t^3}{6\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right)\frac{t^2}{2}\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]t\right\}\right]$$
$$= \frac{\mu_4}{24\mu_1^2} - \frac{\mu_2\mu_3}{6\mu_1^3} + \frac{\mu_2^3}{8\mu_1^4}. \tag{1.24}$$

Provided that $\mu_3 = E(X^2) < \infty$ and $\mu_4 = E(X^2) < \infty$.

Proof. Please see Appendix A4.

In general, the distribution of $H(\cdot)$, although not known, has smaller variance compared to the initial inter-renewal time distribution, $G(\cdot)$, and consequently has a rather small coefficient of variation. Please note that $H(\cdot)$, has the same variance as the excess life for the renewal process of $N_2(y)$ beyond Q_D. For example, for the initial renewal process given in Table 1.2 with $c_X^2 = 0.5$, we observe that $H(\cdot)$ has $c_X^2 = 0.05$ and the reason can be explained by Equations (1.20) and (1.21). This phenomenon is considered to be a disadvantage in terms of approximating the renewal function and its integral for the renewal process associated with $H(\cdot)$ since convergence of the asymptotic approximations for $M_H(t)$ happens at relatively larger values of t based on Equation (1.7). Furthermore, the values of the renewal function $M_H(t)$ and its integral keep zero from $t = 0$ until $t = Q_D$. This in return, is an advantage for the modified approximation described in Section 3.2 which keeps the values of the renewal function and its integral at zero for small values of t.

We compare the results using three measures of Δ_1, Δ_2 used in the study of Cetinkaya et al. (2008) and Δ_3, where

$$\Delta_1\% = \frac{|C^* - C^s(Q_R^s, Q_D^s)|}{C^s(Q_R^s, Q_D^s)} \times 100, \tag{1.25}$$

$$\Delta_2\% = \frac{|C^s(Q_R^*, Q_D^*) - C^s(Q_R^s, Q_D^s)|}{C^s(Q_R^s, Q_D^s)} \times 100, \tag{1.26}$$

$$\Delta_3\% = \frac{|C^* - C^s(Q_R^*, Q_D^*)|}{C^s(Q_R^*, Q_D^*)} \times 100 \tag{1.27}$$

In the Equations (1.25) through (1.27) superscript * indicates that the values of the variables are obtained by minimizing the cost function of (1.19) and superscript ^{S} indicates that the values of the variables are obtained by simulation. C^* denotes the calculated optimum cost obtained by minimizing approximated Equation (1.19). The simulated cost of the optimum policy obtained by minimizing approximated Equation (1.19), is denoted by $C^s(Q_R^*, Q_D^*)$ and the optimum cost obtained by simulation (enumeration) is denoted by $C^s(Q_R^s, Q_D^s)$. Among the three measures, $\Delta_2\%$ is the most important because it measures the actual performance of a policy that is obtained based on the selected approximations.

The asymptotic approximation defined in Equations (1.5) and (1.6) to minimize Equation (1.19) sometimes results in smaller calculated cost compared to the modified approximation. The reason is the negative renewal function values at small t (here Q_D) and large values of integral of renewal function values at small t due to the negatively signed integral of renewal function in Equation (1.19) (Please see Appendix A.3 for more details). Although smaller calculated cost value of a policy seems to be favorable, it is not achievable practically and different from the actual cost of implementing the policy.

Table 1.9 Policy and cost comparison: asymptotic versus modified approximation*

A_R	h	A_D	w	λ	Q_D^*	Q_R^*	C^*	$\Delta_1\%$	$\Delta_2\%$	$\Delta_3\%$
Parameters					*Asymptotic approximation*					
80	1	10	4	1	3	24	37.08	0.58%	0.29%	0.87%
80	1	40	4	1	7	23	53.98	1.86%	0.31%	2.17%
80	1	40	8	1	4	24	60.03	0.98%	0.03%	1.01%
80	1	80	16	0.1	13	82	284.12	0.85%	0.13%	0.99%
80	4	10	16	0.1	6	40	251.57	1.19%	0.49%	1.67%
320	1	10	4	1	3	52	65.41	0.45%	0.13%	0.58%
320	1	40	8	0.1	4	52	88.29	0.81%	0.07%	0.86%
320	4	10	8	0.1	11	82	410.39	2.12%	0.56%	2.66%
320	4	40	16	0.1	14	81	540.35	2.38%	0.59%	2.98%
320	4	40	16	1	3	24	148.3	0.47%	0.42%	0.88%
Parameters					*Modified approximation*					
A_R	h	A_D	w	λ	Q_D^*	Q_R^*	C^*	$\Delta_1\%$	$\Delta_2\%$	$\Delta_3\%$
80	1	10	4	1	2	24	37.32	0.05%	0.03%	0.02%
80	1	40	4	1	7	22	53.13	0.23%	0.08%	0.14%
80	1	40	8	1	4	24	60.72	0.16%	0.03%	0.13%
80	1	80	16	0.1	12	81	287.83	0.44%	0.05%	0.39%
80	4	10	16	0.1	5	40	255.42	0.33%	0.02%	0.30%
320	1	10	4	1	2	53	65.70	0.00%	0.06%	0.06%
320	1	40	8	0.1	4	53	89.11	0.11%	0.05%	0.04%
320	4	10	8	0.1	8	83	420.63	0.32%	0.05%	0.27%
320	4	40	16	0.1	12	81	555.70	0.39%	0.06%	0.33%
320	4	40	16	1	2	24	149.26	0.17%	0.15%	0.02%

* Demand follows Erlang distribution (2, 2.5)

Consider a policy that is obtained by applying asymptotic approximation, e.g. $C^*(3,24) = 148.3$. The simulation result shows that if we put this policy into practice, the system yields the cost of $C^s(3,24) = 149.62$ with a deviation of 0.47% compared to

0.17% when the policy of (2, 24) from the modified approximation is applied. Other values of accuracy measures are shown in Table 1.9 for the instances with small Q_D values. Please note that the integral for $M_H(t)$ in modified approximation is zero from $t = 0$ to $t = Q_D$ for any Q_D value (Please see Appendix A.3 for more details)

1.6 Summary and conclusion

In this chapter we introduced a simple but effective approximation for the renewal function and its integral. The commonly used approximations of the renewal function and its integral based on asymptotic behaviors do not perform well for small values of t. The main focus of this research is to improve the performance of the approximation when the values of t are outside the typical asymptotic range. By plotting and studying the simulated values of renewal function and renewal function integral, we developed a three-piece function with two switch-over points that resembles the simulation results of renewal function and its integral. Intensive numerical studies show that our modified approximation outperforms the asymptotic approximation. The modified approximation is easy to implement, especially useful for decision makings. A comprehensive application case from the literature is used to demonstrate the applicability and performance of the proposed modified approximation. In the future, more applications will be investigated to test the effectiveness of the proposed modified approximations.

1.7 References

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CHAPTER II

A STOCHASTIC PROCESS STUDY OF TWO-ECHELON SUPPLY CHAIN INVENTORY DECISIONS WITH BULKY DEMAND

2.1 Introduction and literature review

This chapter compares the centralized and decentralized systems for a supply chain comprised of a vendor and a retailer, who faces bulky demands. Customers arrive as a renewal process and the amount of each customer order follows an independent and identical distribution. Some literature of supply chain management assumes policies are set by a central decision maker to optimize total supply chain performance (Lee and Whang 1999) while some other research studies the decentralized case (e.g. Cachon and Zipkin 1999, Lee and Whang 1999, Chen et al. 2001, Porteus 2000). As a centralized system, the benefits of Vendor-Managed Inventory (VMI) in a supply chain have been well recognized by numerous success stories in the retail industry. A VMI supplier has the right of controlling the downstream resupply decision rather than just filling orders placed by downstream players. On the other hand, many supply chains in practice still operate in a decentralized mode, in which each business entity is responsible for its own inventory policy decisions (Horngren and Foster 1991) based on individual entity performance. Lee and Whang (1999) assumed that the most downstream echelon is charged for all backorder penalties and the upstream echelon is charged for their holding cost. Porteos (2000) offered a scheme called responsibility tokens that endow the system with a self-correcting property. The game theory is often involved in studying decentralized systems. Cachon and Zipkin (1999), for example, studied the echelon inventory game and the local inventory game. In their study the parties across two echelons played a Nash equilibrium. Many studies investigated how a supplier could induce a retailer to behave in a manner that is more favorable to the supplier (Donohue 2000, Tsay 1999, Ha 2001, Lal and Staelin 1984, Moses and Seshadri 2000, Kraiselburd et al. 2004, Pasternack 1985). Chen (1999) studied competitive selection of inventory policies in a multi-echelon model with deterministic demand.

This study considers the case in which both the customer order arrivals and sizes are stochastic as a general stochastic processes, similar to the problem setting used by Cetinkaya et al. (2008). The compound Poisson process has been widely used to model demand process in the inventory management literature. Chen (1998), for example, considered discrete distributed demand size with a Poisson customer arrival process to study the value of demand information sharing in a supply chain. It has been also shown that inventory coordination can help save costs for the case of compound Poisson demand and zero lead time (Thompstonej and Silver 1975). This study will relax the assumption of the Poisson arrivals of customers and use a general distribution to model the inter-arrival times of customers. In order to attack the non-Poisson arrivals, the renewal function will be used in the analysis (Tijms 2003).

2.2 Problem statement

Consider a two-echelon supply chain including a vendor and a retailer, both implementing a base-stock inventory policy. All the demands must be satisfied and it is assumed that the order lead-times are negligible. Both the vendor and the retailer follow a quantity-based policy and place an order immediately after their inventory goes below zero. The costs in the system include A_r, fixed cost of shipment from the vendor to the retailer, A_v, fixed cost of shipment to the vendor, h_r, unit inventory holding cost for the supplier, and h_v, unit inventory holding cost for the vendor. In the decentralized case, the retailer is responsible for A_r and h_r while the vendor is responsible for A_v and h_v. Successive orders from customers reach the retailer following a stochastic process with inter-arrival times T_n. The stochastic process $\{T_n, n = 1,2, \dots\}$ consists of independent and identically distributed nonnegative random variables with finite mean $1/\lambda$. The orders are considered bulk where the size of the n^{th} order from customers to the retailer is denoted by Y_n. The stochastic process $\{Y_n, n = 1,2, \dots\}$ also consists of independent and identically distributed nonnegative random variables with distribution $G(\cdot)$ with finite mean μ_1. The inventory decision variables are Q_r as the base stock of the retailer and Q_v as the base stock of vendor. The time at which n^{th} customer order reaches the retailer is $S_n = \sum_{i=1}^{n} T_i$, $n \geq 1$, $S_0 = 0$ and therefore the number of retailer's order arrivals up to time *t* is defined by $N_1(t) = \sup\{n: S_n \leq t\}$. The accumulated demand immediately after the n^{th} customer order is $D_n = \sum_{i=1}^{n} Y_i$ and the number of customer order to accumulate the total demand just beyond Q_r is $N_2(Q_r) + 1 = \inf\{n: D_n > Q_r\}$ where $N_2(y) = \sup\{n: D_n \leq y\}$. By assuming the two stochastic processes of $\{T_n\}$ and $\{Y_n\}$ are independent, we may face three inventory management cases based on the values of Q_v and Q_r. First, if $Q_v > Q_r > 0$, then each replenishment of the vendor consists of more than one replenishment of retailer. Second, if $Q_v \leq Q_r$, then each replenishment of the retailer triggers one replenishment of vendor so that Q_v will be 0 to eliminate the vendor's inventory holding cost without changing the vendor's ordering costs. The final case is that $Q_r = 0$ while $Q_v > 0$, in which the retailer does not hold any inventory and places an order with the vendor when seeing a customer order.

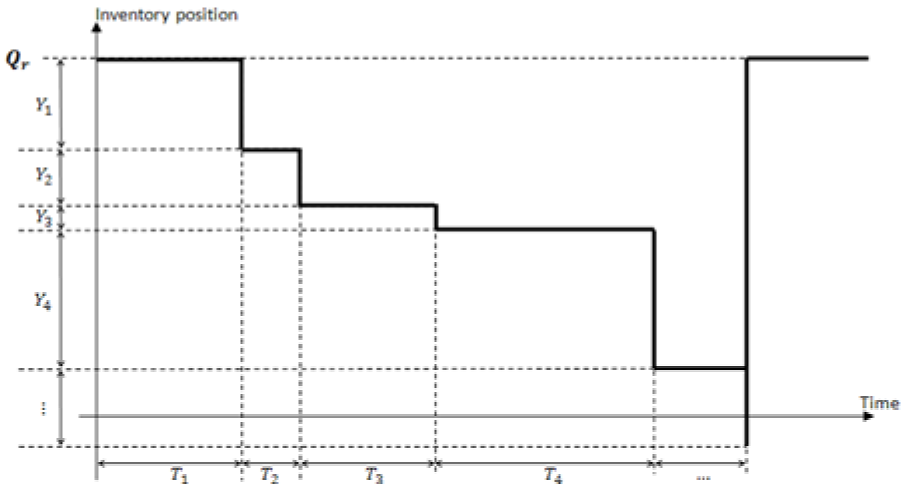

Figure 2.1 A replenishment cycle at the retailer

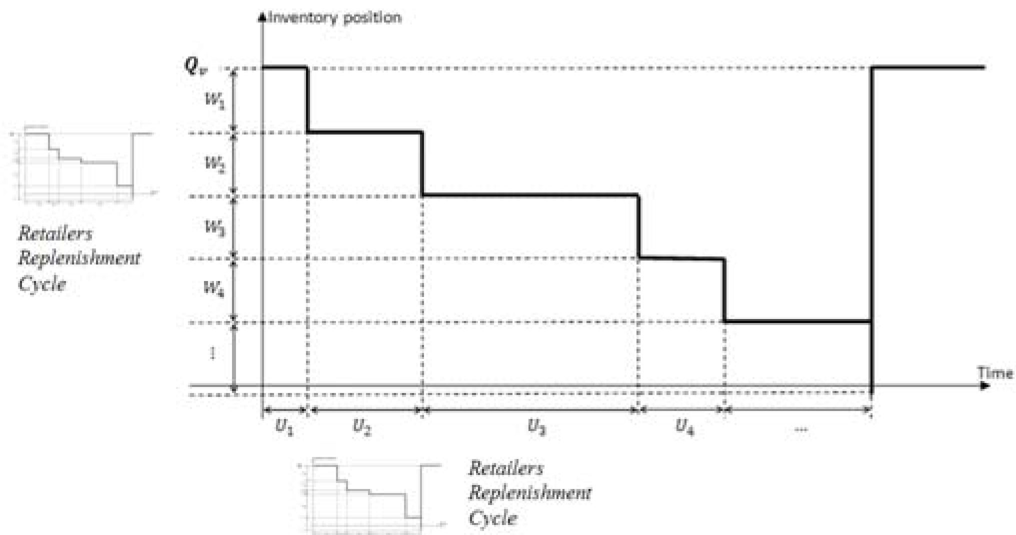

Figure 2.2 A replenishment cycle at the vendor

2.3 Problem formulation and analysis

The inventory process for the whole system is a regenerative process and regeneration happens when the vendor replenishes her inventory. Therefore, the cycle for the overall system is defined as the replenishment cycle of the vendor. In each cycle, the retailer may replenish her inventory once or more. Figure 2.1 shows a sample replenishment cycle at the retailer and Figure 2.2 illustrates a cycle that happens at the vendor for the case in which $Q_v > Q_r > 0$. Let CC_n denote the total cost incurred at both parties during the n^{th} cycle length and CL_n denote the n^{th} cycle length for the overall system. Based on the Renewal-Reward theorem, the expected long-run average total cost per time unit is $C(Q_v, Q_r) = \frac{E(CC_n)}{E(CL_n)}$ for any n. Let U_i denote the i^{th} replenishment cycle length at the retailer and W_i is the cumulated customer orders during the i^{th} retailer replenishment cycle. Under our assumptions, processes $\{U_i\,, i = 1,2, \dots\}$ and $\{W_i\,, i = 1,2, \dots\}$ are independent and identically distributed nonnegative random variables as

$$U_i \sim S_{N_2(Q_r)+1} = \sum_{n=1}^{N_2(Q_r)+1} T_n\,, \text{and} \tag{2.1}$$

$$W_i \sim D_{N_2(Q_r)+1} = \sum_{n=1}^{N_2(Q_r)+1} Y_n. \tag{2.2}$$

Therefore,

$$E[U_i] = E\left[S_{N_2(Q_r)+1}\right] = \frac{1}{\lambda} E[N_2(Q_r) + 1] = \frac{1}{\lambda}(M_G(Q_r) + 1), \text{and} \tag{2.3}$$

$$E[W_i] = E\left[D_{N_2(Q_r)+1}\right] = \mu_1 E[N_2(Q_r) + 1] = \mu_1(M_G(Q_r) + 1), \tag{2.4}$$

where $M_G(\cdot)$ is the renewal function for $G(\cdot)$, which characterizes the renewal process of $\{Y_n\,, n = 1,2, \dots\}$.

2.3.1 Expected cycle length

Based on the definition, each cycle (between two consecutive replenishments at the vendor) consists of at least one retailer replenishment, so $E[CL_n] = E(\sum_{i=1}^{K} U_i)$, where K denotes the number of retailer replenishments in one cycle and $K =$

$\inf\{k: \sum_{i=1}^{k} W_i > Q_v\}$. Having $H(.)$ as the distribution function of W_i, $H^{(k)}(\cdot)$ denotes the k-fold convolution of $H(\cdot)$. Therefore, $Prob(K \geq k) = H^{(k-1)}(Q_v)$ and $E[K] = \sum_{k=1}^{\infty} Prob(K \geq k)$, so

$$E[K] = \sum_{k=1}^{\infty} H^{(k-1)}(Q_v) = M_H(Q_v) + 1, \tag{2.5}$$

where $M_H(.)$ denotes the renewal function of $H(\cdot)$. Based on the Wald's equation (Tijms 2003), we have

$$E[CL_n] = \frac{1}{\lambda} E(M_G(Q_r) + 1)E[K] = \frac{1}{\lambda}(M_G(Q_r) + 1)(M_H(Q_v) + 1). \tag{2.6}$$

2.3.2 Expected cycle cost

The cost of each cycle consists of two components: inventory holding cost and shipment cost for both the retailer and the vendor. The area below the inventory line in Figure 2.1 times the retailer's unit inventory holding cost times the number of retailer replenishments in one cycle calculates the total holding cost of the retailer in one cycle as

$$E(RH) = h_r \frac{1}{\lambda}(M_H(Q_v) + 1)\left(Q_r + \int_0^{Q_r} (Q_r - y) dM_G(y)\right) \tag{2.7}$$

Similarly the area below the inventory line in Figure 2.2 times the vendor's unit inventory holing cost calculates the total holding cost of the vendor in one cycle, which is,

$$E(VH) = h_v \frac{1}{\lambda}(M_G(Q_r) + 1)\left(Q_v + \int_{Q_r}^{Q_v} (Q_v - w) dM_H(w)\right). \tag{2.8}$$

Since the cycle length is defined as one vendor's replenishment cycle, the shipment cost to the vendor is $E(VS) = A_v$ and the shipment cost to the retailer is $E(RS) = A_r E[K] = A_r(M_H(Q_v) + 1)$. The total cost in one system-wide cycle is then $E(CC_n) = E(RH) + E(VH) + E(VS) + E(RS)$.

2.3.3 Expected long-run average cost

Based on the Renewal-Reward theorem, the expected long-run average total cost per time unit is $C(Q_v, Q_r) = \frac{E(CC_n)}{E(CL_n)}$ for any n and

$$C(Q_v, Q_r) = \frac{h_r\left(Q_r + \int_0^{Q_r}(Q_r - y)dM_G(y)\right) + A_r\lambda}{M_G(Q_r) + 1} + \frac{h_v(M_G(Q_r) + 1)\left(Q_v + \int_{Q_r}^{Q_v}(Q_v - w)dM_H(w)\right) + A_v\lambda}{(M_G(Q_r) + 1)(M_H(Q_v) + 1)}. \tag{2.9}$$

The optimal inventory policy for the centralized system is $(Q_v^*, Q_r^*) = \text{argmin}_{Q_v\geq 0, Q_r\geq 0}\, C(Q_v, Q_r)$. Please note that when $Q_v < Q_r$, Equation (2.9) will be reduced to $\frac{h_r\left(Q_r+\int_0^{Q_r}(Q_r-y)dM_G(y)\right)+A_r\lambda}{M_G(Q_r)+1} + \frac{A_v\lambda}{M_G(Q_r)+1}$ because $Q_v = 0$.

In the decentralized case, each party is responsible for her own cost and the two parties make their decisions in a sequential way. The retailer's long-term average total cost per time unit is $C^r(Q_r) = \frac{h_r\frac{1}{}\left(Q_r+\int_0^{Q_r}(Q_r-y)dM_G(y)\right)+A_r}{\frac{1}{}(M_G(Q_r)+1)}$, which is the first term in Equation (2.9).

The optimal policy for the retailer in the decentralized case is decided by $Q_r^d = \text{argmin}_{\,Q_r\geq 0}\, C^r(Q_r)$. After knowing Q_r^d, the vendor minimizes her cost $C^v(Q_v) = \frac{h_v\frac{1}{}(M_G(Q_r^d)+1)\left(Q_v+\int_{Q_r^d}^{Q_v}(Q_v-w)dM_H(w)\right)+A_v}{\frac{1}{}(M_G(Q_r^d)+1)(M_H(Q_v)+1)}$, which is the second term in Equation (2.9), to get her optimal policy as $Q_v^d = \text{argmin}_{\,Q_v\geq 0}\, C^v(Q_v)$.

2.4 Approximation

Equation (2.9) is too complicated to be optimized directly, we propose approximations for $M_G(Q_r)$, $\int_0^{Q_r} M_G(y)\, dy$, $M_H(Q_v)$, and $\int_0^{Q_v} M_H(w)\, dw$ based on the results of chapter one. We approximate the cost function in three regions of Q_r and Q_v. In the first region, the renewal function and its integral have the value of zero. In the second region, we apply the modified approximations and in the third region we use the asymptotic expansions to approximate the cost function. Comparing the minimum costs in the three regions lead to the global minimum cost and the corresponding optimal policy. The approximations for the third region is as follows,

$$M_G(Q_r) \approx \frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}, \tag{2.10}$$

$$\int_0^{Q_r} M_G(Q_r)dr \approx \frac{{Q_r}^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right) Q_r + \frac{{\mu_2}^2}{4{\mu_1}^3} - \frac{\mu_3}{6{\mu_1}^2}, \tag{2.11}$$

$$M_H(Q_v) \approx \frac{Q_v}{\vartheta_1} + \frac{\vartheta_2 - 2{\vartheta_1}^2}{2{\vartheta_1}^2}, and \tag{2.12}$$

$$\int_0^{Q_v} M_H(w)dw \approx \frac{{Q_v}^2}{2\vartheta_1} + \left(\frac{\vartheta_2}{2{\vartheta_1}^2} - 1\right) Q_v + \frac{{\vartheta_2}^2}{4{\vartheta_1}^3} - \frac{\vartheta_3}{6{\vartheta_1}^2}. \tag{2.13}$$

Here, μ_1, μ_2 and μ_3 denote the first, second and third moments of $G(\cdot)$ and ϑ_1, ϑ_2 and ϑ_3 denote the first, second and third moments of $H(\cdot)$ respectively. For the moments of $H(\cdot)$, we have

$$\vartheta_1 = \mu_1(M_G(Q_r) + 1), \tag{2.14}$$

$$\vartheta_2 = \mu_2(M_G(Q_r) + 1) + 2\mu_1\left(Q_r M_G(Q_r) - \int_0^{Q_r} M_G(y)dy\right), \text{and} \tag{2.15}$$

$$\vartheta_3 = \mu_3 + 3\mu_1 \int_0^{Q_r} y^2 m_G(y)dy + 3\mu_2 \int_0^{Q_r} y m_G(y)dy + \mu_3 \int_0^{Q_r} m_G(y)dy. \tag{2.16}$$

In order to obtain ϑ_3, we need to apply the following approximation for $\int_0^{Q_r} y^2 m_G(y)dy$.

$$\int_0^{Q_r}\int_0^{y} M_G(x)dx\,dy$$
$$\approx \left[\frac{{Q_r}^3}{6\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right)\frac{{Q_r}^2}{2}\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] Q_r + \frac{\mu_4}{24\mu_1^2} - \frac{\mu_3\mu_2}{6\mu_1^3} + \frac{\mu_2^3}{8\mu_1^4}. \tag{2.17}$$

(2.10), (2.12), (2.14), and (2.15) are only used in the study of Cetinkaya et al. (2008) to approximate their cost functions. Their approximation was verified by simulation for exponentially distributed or Erlang-*k* distributed demand quantities. We suspect that application of their approximations to $\int_0^{Q_v} M_H(w)dw$ may introduce more errors when the customer order quantities follow other distributions. Therefore, we introduce the approximations of (2.11) (see Tijms (2003), (2.13), (2.16), and (2.17) to obtain the approximate total cost function for the centralized case as

$$C'^{(Q_v,Q_r)} = \frac{h_r\left(\frac{{Q_r}^2}{2\mu_1} + \frac{\mu_2 Q_r}{2{\mu_1}^2} + \frac{{\mu_2}^2}{4{\mu_1}^3} - \frac{\mu_3}{6{\mu_1}^2}\right) + \lambda A_r}{\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)} + \frac{h_v\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)\left(\frac{{Q_v}^2}{2\vartheta_1} + \frac{\vartheta_2 Q_v}{2{\vartheta_1}^2} + \frac{{\vartheta_2}^2}{4{\vartheta_1}^3} - \frac{\vartheta_3}{6{\vartheta_1}^2}\right) + \lambda A_v}{\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)\left(\frac{Q_v}{\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}\right)}. \tag{2.18}$$

The optimal inventory policy for the centralized system is

$$({Q'}^*_v, {Q'}^*_r) = \underset{Q_v \geq Q_r \geq 0 \text{ or } Q_r \geq Q_v = 0}{\operatorname{argmin}} C'(Q_v, Q_r). \tag{2.19}$$

In the decentralized case, the retailer's long-term average cost per time unit is approximated by $C'^r(Q_r) = \frac{h_r\left(\frac{{Q_r}^2}{2\mu_1} + \frac{{}_2 Q_r}{2{\mu_1}^2} + \frac{{}_2{}^2}{4{\mu_1}^3} - \frac{{}_3}{6{\mu_1}^2}\right) + \lambda A_r}{\left(\frac{Q_r}{{}_1} + \frac{{}_2}{2{\mu_1}^2}\right)}$ and the optimal policy for the retailer has a close form of $\mathrm{Q'}^{\mathrm{d}}_{\mathrm{r}} = \max\left\{\frac{-3\mathrm{h_r}\mu_2 + \sqrt{3}\sqrt{24\mathrm{A_r}\mathrm{h_r}\lambda\mu_1{}^3 + 3\mathrm{h_r}{}^2\mu_2{}^2 - 4\mathrm{h_r}{}^2\mu_1\mu_3}}{6\mathrm{h_r}\mu_1}, 0\right\}$.

The vendor's long-term average cost per time unit is approximated by $C'^v(Q_v) = \frac{h_v\left(\frac{{Q'}^d_r}{{}_1} + \frac{{}_2}{2{\mu_1}^2}\right)\left(\frac{{Q_v}^2}{2\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}Q_v + \frac{{\vartheta_2}^2}{4{\vartheta_1}^3} - \frac{\vartheta_3}{6{\vartheta_1}^2}\right) + \lambda A_v}{\left(\frac{{Q'}^d_r}{{}_1} + \frac{{}_2}{2{\mu_1}^2}\right)\left(\frac{Q_v}{\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}\right)}$ and the optimal policy for the vendor is decided by $\operatorname{argmin}_{Q_v \geq 0} C'^v(Q_v)$, which has the close form of

$$\frac{-3\vartheta_2 h_v[\frac{\mu_2}{2\mu_1^2}+\frac{Q'^d_r}{\mu_1}]+\sqrt{3}\sqrt{24\lambda A_v\vartheta_1^3 h_v[\frac{\mu_2}{2\mu_1^2}+\frac{Q'^d_r}{\mu_1}]+3\vartheta_2^2 h_v[\frac{\mu_2}{2\mu_1^2}+\frac{Q'^d_r}{\mu_1}]^2-4\vartheta_1\vartheta_3 h_v[\frac{\mu_2}{2\mu_1^2}+\frac{Q'^d_r}{\mu_1}]^2}}{6\vartheta_1 h_v[\frac{\mu_2}{2\mu_1^2}+\frac{Q'^d_r}{\mu_1}]}. \tag{2.20}$$

The vendor's optimal policy under the approximation is given by solving

$$Q'^d_v = \begin{cases} \operatorname*{argmin}_{Q_v\geq 0} C'^v(Q_v) & \text{If } \operatorname*{argmin}_{Q_v\geq 0} C'^v(Q_v) \geq Q'^d_r \\ \operatorname*{argmin}_{Q_v=0 \text{ or } Q'^d_r} C'^v(Q_v) & \text{Otherwise.} \end{cases} \tag{2.21}$$

2.5 Numerical experiments

In order to demonstrate the analysis procedure laid out in Section 2.4 and evaluate the benefit of centralization, such as VMI, comprehensive numerical experiments with the parameter values in Table 2.1, same as those in the study of Cetinkaya et al. (2008) for their one-player's case, are conducted. There are 1,024 instances for all combinations of parameters' values. The demand inter-arrival is assumed to follow an exponential distribution.

Table 2.1 Parameters' values used in numerical experiments

A_v	h_v	A_r	h_r	λ	μ_1
40	1	5	2	1	5
80	2	10	4	10	20
160	4	20	8		
320	8	40	16		

Tables 2.2 summarizes the average savings under different ratios of A_v/A_r and h_v/h_r. Figure 2.3 also illustrates the changes in saving separately for A_v/A_r and h_v/h_r

ratios. In general, the savings are more significant when h_r is relatively small compared to h_v. The reason can be explained as that by increasing h_v , assuming fixed A_v, A_r and h_r, the vendor tends to shift the inventory to the retailer where the inventory holding cost is not increased. This is what happens in the centralized system to minimize the system-wide cost. But in a decentralized system it is too late for the vendor to shift the inventory to the retailer's side since the retailer has already minimized her cost and determined the inventory decisions. The reason of changes in saving when the A_v/A_r ratio is more complicated in a way that it relates to the inventory holding costs as well. Supposing fixed costs of A_r, h_v and h_r, by increasing A_v the vendor tries to order less frequently or more at a time. This is the only option for the vendor in a decentralized system. But in the centralized system, the vendor has another important option specially when the vendor's inventory holding cost, h_v, is relatively high and the retailers costs are relatively small.

Table 2.2 Average savings caused by centralization for all cases

	h_v/h_r						
A_v/A_r	0.0625	0.125	0.25	0.5	1	2	4
1	0.00%	0.00%	2.41%	4.18%	4.91%	5.08%	5.88%
2	0.00%	0.00%	0.00%	6.57%	11.28%	12.17%	12.95%
4	0.00%	0.00%	0.00%	2.69%	16.83%	23.03%	25.03%
8	0.00%	0.00%	0.00%	0.00%	13.67%	31.17%	37.55%
16	0.00%	0.00%	0.00%	0.00%	10.47%	31.27%	47.12%
32	0.00%	0.00%	0.00%	0.00%	7.52%	31.41%	49.26%
64	0.00%	0.00%	0.00%	0.00%	5.16%	31.60%	50.75%

So in this case the vendor's option in the centralized system is to shift the inventory to the retailer's side and benefit from low costs of the retailer; the strategy that is not possible in the decentralized system. This is the case where a significant saving can be observed. But even in centralized case and where the costs of the retailer are also high, the vendor just considers a tradeoff between her own inventory holding and fixed costs to determine her inventory decision. This case is not expected to result in a significant saving.

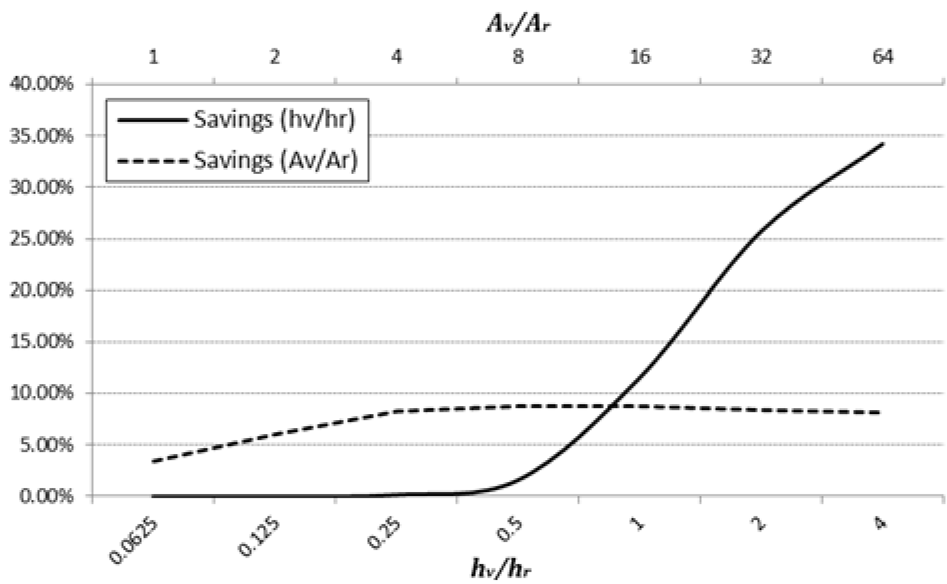

Figure 2.3 Average savings caused by centralization for all cases

Tables 2.3 and 2.4 show in detail how changes in inventory holding and fixed costs of the retailer affect the retailer's and the vendor's inventory policy decisions in a relative slow moving and fast moving consumer demand processes. As both tables how, by increasing the retailer's inventory holding cost the retailer's order quantity decreases. And by increasing the retailer's fixed cost the retailer's order quantity increases to escape from high inventory holding cost charges. Table 2.5 also shows how the vendor's inventory decisions are affected by changes in her inventory holding and fixed costs. As it is expected by increasing the vendor's inventory holding cost, the vendor's order quantity decreases and by increasing the vendor's fixed cost, her order quantity increases. Please note that unlike the Tables 2.3 and 2.4 and since the retailer's costs are fixed, the retailer's order quantities of decentralized case are all the same in Table 2.5.

Table 2.3 Sample centralization saving with $\lambda = 10$, $\mu_1 = 5$, A_r, and h_r

A_v	h_v	A_r	h_r	$Q_r'^d$	$Q_v'^d$	$C'^r(Q_r'^d) + C'^v(Q_v'^d)$	$Q_r'^*$	$Q_v'^*$	$C'(Q_r'^*, Q_v'^*)$	$Saving$
320	2	5	2	12	118	283.699	123	0	254.87	10.16%
			4	7	120	295.571	7	120	295.57	0.00%
			8	0	123	302.9	0	123	302.9	0.00%
			16	0	123	302.9	0	123	302.9	0.00%
		10	2	18	114	296.866	125	0	256.82	13.49%
			4	12	118	314.627	12	118	314.62	0.00%
			8	7	120	338.292	7	120	338.29	0.00%
			16	0	123	352.9	0	123	352.9	0.00%
		20	2	28	110	315.172	126	0	260.69	17.29%
			4	18	114	341.101	19	115	341.09	0.00%
			8	12	118	376.48	12	119	376.49	0.00%
			16	7	120	423.73	7	120	423.74	0.00%
		40	2	41	103	340.808	130	0	268.25	21.29%
			4	28	110	378.07	28	110	378.07	0.00%
			8	18	114	429.57	18	115	429.56	0.00%
			16	12	118	500.2	12	118	500.2	0.00%

Tables 2.6 and 2.7 shows how the inventory policy decisions of the vendor and the retailer are affected by the parties fixed cost and inventory holding cost respectively. Table 2.6 shows that by increasing the vendors fixed cost her order quantity increases. Also by increasing the retailer's fixed cost the retailer's order quantity increases in both

decentralized and centralized systems and that of the vendor's decreases in decentralized case.

Table 2.4 Sample centralization saving with $\lambda = 10$, $\mu_1 = 20$, A_r, and h_r

A_v	h_v	A_r	h_r	$Q_r'^d$	$Q_v'^d$	$C'^r(Q_r'^d) + C'^v(Q_v'^d)$	$Q_r'^*$	$Q_v'^*$	$C'(Q_r'^*, Q_v'^*)$	$Saving$
320	8	5	2	0	111	1056.4	239	0	509.22	51.80%
			4	0	111	1056.4	164	0	719.17	31.92%
			8	0	111	1056.4	112	0	1014.3	3.99%
			16	0	111	1056.4	0	111	1056.4	0.00%
		10	2	28	102	1088.942	241	0	513.13	52.88%
			4	0	111	1106.4	166	0	724.71	34.50%
			8	0	111	1106.4	112	0	1022.2	7.61%
			16	0	111	1106.4	0	111	1106.4	0.00%
		20	2	47	93	1122.07	246	0	520.87	53.58%
			4	28	102	1174.38	169	0	735.66	37.36%
			8	0	111	1206.4	115	0	1037.7	13.98%
			16	0	111	1206.4	0	111	1206.4	0.00%
		40	2	73	0	904.193	254	0	536.01	40.72%
			4	47	93	1245.77	174	0	757.1	39.23%
			8	28	102	1345.27	118	0	1068.1	20.60%
			16	0	111	1406.4	0	111	1406.4	0.00%

Table 2.5 Sample centralization saving with $\lambda = 1$, $\mu_1 = 5$, h_r, and h_v

A_r	h_r	A_v	h_v	$Q_r'^d$	$Q_v'^d$	$C'^r(Q_r'^d) + C'^v(Q_v'^d)$	$Q_r'^*$	$Q_v'^*$	$C'(Q_r'^*, Q_v'^*)$	$Saving$
5	2	40	1	0	16	24.725	0	16	24.725	0.00%
			2	0	10	32.5	11	0	29.263	9.96%
			4	0	6	42.769	11	0	29.263	31.58%
			8	0	0	45	11	0	29.263	34.97%
		80	1	0	24	33.092	0	24	33.092	0.00%
			2	0	16	44.449	17	0	40.705	8.42%
			4	0	10	60	17	0	40.705	32.16%
			8	0	6	80.538	17	0	40.705	49.46%
		160	1	0	36	44.863	0	36	44.863	0.00%
			2	0	24	61.185	25	0	57.065	6.73%
			4	0	16	83.899	25	0	57.065	31.98%
			8	0	10	115	25	0	57.065	50.38%
		320	1	0	53	61.472	0	53	61.472	0.00%
			2	0	36	84.726	36	0	80.355	5.16%
			4	0	24	117.37	36	0	80.355	31.54%
			8	0	16	162.8	36	0	80.355	50.64%

Table 2.6 Sample centralization saving with $\lambda = 10$, $\mu_1 = 20$, A_r, and A_v

h_v	h_r	A_r	A_v	$Q_r'^d$	$Q_v'^d$	$C'^r(Q_r'^d) + C'^v(Q_v'^d)$	$Q_r'^*$	$Q_v'^*$	$C'(Q_r'^*, Q_v'^*)$	$Saving$
2	2	5	40	0	74	226.92	79	0	187.88	17.20%
			80	0	111	301.6	115	0	259.42	13.99%
			160	0	163	406.79	166	0	362.35	10.92%
			320	0	236	555.28	239	0	509.22	8.29%
		10	40	28	64	261.322	84	0	198.24	24.14%
			80	28	102	336.312	119	0	267.02	20.60%
			160	28	154	441.722	169	0	367.83	16.73%
			320	28	228	590.352	241	0	513.13	13.08%
		20	40	47	0	252.722	94	0	217.49	13.94%
			80	47	93	373.28	126	0	281.6	24.56%
			160	47	146	479.07	174	0	378.55	20.98%
			320	47	219	627.97	246	0	520.87	17.05%
		40	40	73	0	267.8291	111	0	251.6	6.06%
			80	73	0	358.738	140	0	308.71	13.95%
			160	73	132	530.5	185	0	399.13	24.76%
			320	73	206	679.93	254	0	536.01	21.17%

Table 2.7 Sample centralization saving with $\lambda = 10$, $\mu_1 = 20$, h_r, and h_v

A_v	A_r	h_r	h_v	$Q_r'^d$	$Q_v'^d$	$C'^r(Q_r'^d) + C'^v(Q_v'^d)$	$Q_r'^*$	$Q_v'^*$	$C'(Q_r'^*, Q_v'^*)$	$Saving$
40	10	2	1	28	102	210.872	84	0	198.24	5.99%
			2	28	64	261.322	84	0	198.24	24.14%
			4	28	0	271.489	84	0	198.24	26.98%
			8	28	0	271.489	84	0	198.24	26.98%
		4	1	0	111	225.8	0	111	225.8	0.00%
			2	0	74	276.92	0	74	276.92	0.00%
			4	0	47	347.39	54	0	277.86	20.01%
			8	0	28	441.77	54	0	277.86	37.10%
		8	1	0	111	225.8	0	111	225.8	0.00%
			2	0	74	276.92	0	74	276.92	0.00%
			4	0	47	347.39	0	47	347.39	0.00%
			8	0	28	441.77	33	0	385.75	12.68%
		16	1	0	111	225.8	0	111	225.8	0.00%
			2	0	74	276.92	0	74	276.92	0.00%
			4	0	47	347.39	0	47	347.39	0.00%
			8	0	28	441.77	0	28	441.77	0.00%

Table 2.7 illustrates the decrease in order quantity of the vendor as inventory holding cost increases. The savings are also increasing as the vendor's inventory holding cost is increasing since higher inventory holding costs lead to smaller order quantity of

the vendor and larger order quantity of the retailer. Tables 2.6 and 2.7 also explain how the trends in savings may not be monotone as what is expected. For an example consider Table 2.6.

The increase in the vendor's fixed cost results in larger order quantity of the vendor and smaller savings. But as we see in the row where $A_r = 20$ or $A_r = 40$, the saving on the contrary is decreasing and the reason is that the inventory policy pairs are similar i.e. the retailer orders a large quantity while the vendor does not order at all.

In the next section we discuss about the channel inventory and how it has been evolved over time letting the supply chain parties to benefit from the savings caused by altering their inventory management decision. Also we indicate how the shifts in inventory are initiated in different industries as a practical evidence of our findings.

2.6 Discussion on channel inventory

In a conventional supply chain there is no collaboration between the supply chain parties and the buyer determines her order quantity and timing of order placement and is responsible for managing her inventory. The vendor is also responsible for her own operations of ordering and inventory holding. By the growth of information technology, Continues Replenishment Programs (CRP) and Vendor Managed Inventory (VMI) systems appeared to improve supply chain performance. In VMI supply chain systems the vendor decides on the buyer's order quantity and schedule while managing her own operations. In this arrangement the buyer is responsible for her inventory. However, in practice the buyer and the vendor deviate from this arrangement by placing inventory at the buyer under consignment. The vendor may offer the buyer to keep the buyer's stock in consignment. In this case, the vendor holds the ownership of the inventory until they are used by the buyer or delivered to the consumer. Some example of consignment stock can be found in various industries e.g. the high-tech and automotive industries and the industries such as supply medical good, chemicals, construction materials and spare parts. When the vendor owns the buyer's inventory as a consignment stock in a VMI setting, the system is called Vendor-Owned Inventory (VOI). Consignment stock is also possible in conventional supply chain, which is called conventional-plus arrangement (Verheijen 2010). Under VOI, the vendor might share a portion of the cost savings that she gains from knowing the buyer's inventory cost and implementing a VOI system. The buyers always prefer consignments stocks since they are not liable for the risks and do not need to reserve cash for holding the inventory. Therefore, it may be considered as an incentive offer to the buyer in order to place larger order quantities from the vendor. In some industries such as publishing if a vendor wants to remain in business it is viable for her to offer consignment stock. The same situation exist is seasonal product where the buyer does not accept to own the inventory. In some other industries, a vendor initiates the agreement on consignment stock in contrast with the case where the buyer forces the vendor to agree on consignment stock (Verheijen 2010). We investigate the shift in inventory between the buyer and the inventory and its impact on the potential cost savings. The cost structure in general and ratios of the vendors to retailer's inventory holding cost and fixed cost in particular play an important role in determining the inventory decisions of the parties. This concept is the foundation of channel coordination in our study and describes how the parties can benefit from intentionally altering these ratios.

2.7 Conclusion

This study considers a two-echelon supply chain facing a bulk demand process in which the customer order quantities follow a general distribution. After deriving the optimal policy for both the centralized and decentralized cases based on the renewal theory, approximations for the renewal function and its integral are introduced to easily calculate the optimal base stocks. The numerical experiments show that the centralization can lead to significant savings under some circumstances such as the ratio of h_v/h_r is large. We also explain the changes in inventory policy decision of the parties as the cost structure changes as well as different trends in savings caused by changes in cost ratios of the vendor to that of the retailer.

2.8 References

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CHAPTER III

SUPPLY CHAIN COORDINATION OF ORDERING POLICIES INCORPORATING COST SHARING STRATEGIES

This chapter proposes supply chain coordination mechanisms based on cost sharing of either inventory holding costs or ordering costs between the retailer and the vendor in the supply chain described in chapter two.

3.1 Introduction and literature review

The buyer–vendor coordination problem is one of the classical research areas in the multi-echelon inventory literature (e.g., Aysegul and Cetinkaya 2008). A fundamental research stream in this area, known as centralized modeling, recommends integrating and solving the decision problems of the buyer and the vendor together (e.g., Chan et al. 2002, Goyal 1976, Hill 1999, Hoque and Goyal 2000, Lee et al. 2003). Although this approach provides the best result in terms of total system-wide profit/cost, it may not be feasible or desirable by all parties in many practical cases due to incentive conflicts. The alternative approach, known as decentralized modeling, suggests that the retailer and the vendor solve their decision problems independently of each other. However, the total system profits resulting from the centralized approach are superior to those resulting from the corresponding decentralized approach. In other words, decentralized models often result in lost profits for the system when compared to centralized models. As a remedy, another line of research in the literature proposes an alternative approach that relies on using the profit/cost gap between the centralized and decentralized approaches as an inducement to improve decentralized solutions (e.g., Lee and Rosenblatt 1986, Monahan 1984, Taylor 2001). In general, a supply chain is composed of independent firms with individual preferences (Cachon 1999). In contrast to the management of multi-echelon systems, which coordinates inventory, production and distribution decisions at multiple locations of one firm, supply chain management involves coordination of such decisions among multiple and independent firms (Johnson and Pyke 2001). One the major task of supply chain management is to coordinate the processes in the supply chain in such a way, that a given set of objectives is achieved (Stadler 2000). Most commonly, the relevant objectives, pursued by supply chain management, are minimizing system-wide costs while satisfying a predetermined service level (Lee and Billington 1993). The complexity in coordinating the processes in supply chains is introduced by the organizational structure within the network. (Bhatnagar et al. 1993) identify the issue of coordination-at the most general level, which they call general coordination-in integrating decisions of different functions. Within this problem of functional coordination, Bhatnagar et al. (1993) as well as Thomas and Griffin (1996) distinguished three categories of coodinations: (1) supply-production coordination, (2) production-distribution coordination, and (3) inventory-distribution coordination. In this study we will focus on the third category, which is also called buyer–supplier coordination (Thomas and Griffin 1996). For each set of nodes in a supply chain, e.g. a location of a manufacturer and a site of an assembler, a supplier–buyer relationship can be identified (Anupindi and Bassok 1999). Material flows from a supplier to a buyer while information and financial flows are bi-directional. Both in the scientific discussion and in practice considerable attention is paid to the importance of a coordinated relationship between suppliers and buyers. As Goyal and Gupta (1989) noted, coordination between the supplier and the buyer can be mutually beneficial to both. Studies on buyer–supplier coordination have focused on determining the order and production policy that is jointly optimal for both (Sucky 2006). Using such a joint optimal order and production policy-as opposed to independently derived policies- leads to a significant total cost reduction. However, there is an additional set of problems involved in implementing joint policies.

Channel coordination requires the decentralized solution to be improved in a way that (i) it results in the same values for the decision variables as the centralized solution, and (ii) it suggests a mutually agreeable way of sharing the resulting profits. The sharing can be done by means of quantity discounts, rebates, refunds, fixed payments between the parties, and so on. All of these methods represent different forms of incentive schemes, or so-called coordination mechanisms, whose terms can be made explicit under a contract. Consequently, the output of channel coordination, i.e., the coordinated solution, combines the benefits of both centralized and decentralized solutions. However, the above two targets often cannot be both reached, especially when the information is asymmetric.

As an immediate consequence of our analysis in Chapter 2, we observe that the potential cost savings can be obtained by centralization and more importantly, we observe that even a cooperative supply chain model can benefit from savings by changing the cost structures, which lead us to develop coordination strategies.

3.2 Convexity and optimality property for system's cost function

We investigate the convexity of the retailer's and the vendor's cost functions in centralized and decentralized cases to ensure that there exist a unique optimal solution. Equations (3.1) and (3.3) show long-run average cost per time unit of the retailer and the vendor. Equations (3.2) and (3.4) are the derivatives of the cost functions. For functions (3.1) and (3.3) to be convex, the derivative functions (3.2) and (3.4) are required to be increasing functions. Since the behavior and shape of the renewal function, its integral and derivative do not have close forms and are different for different distribution functions. It is technically difficult to investigate the convexity of (3.1) and (3.3) under their current forms.

$$C_r(Q_r) = \frac{h_r\left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + A_r\lambda}{M_G(Q_r) + 1} \tag{3.1}$$

$$\frac{\partial C_v(Q_v)}{\partial Q_r} = h_r - \frac{m_G(Q_r)(h_r\left(Q_r + \int_0^{Q_r} M_G(y)\, dy\right) + A_r\lambda)}{(M_G(Q_r) + 1)^2} \tag{3.2}$$

$$C_v(Q_v) = \frac{h_v(M_G(Q_r) + 1)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v\lambda}{(M_G(Q_r) + 1)(M_H(Q_v) + 1)} \tag{3.3}$$

$$\frac{\partial C_v(Q_v)}{\partial Q_v} = h_v - \frac{m_H(Q_v)(h_v(M_G(Q_r) + 1)(Q_v + \int_0^{Q_v} M_H(w)dw) + A_v\lambda)}{(M_G(Q_r) + 1)(M_H(Q_v) + 1)^2} \tag{3.4}$$

Using the renewal function and its integral we first derive the approximated functions for the three regions where different approximations are used and then investigate the convexity for each region. The three regions include 1) the inventory policy is greater than or equal to the second switchover point, $Q_i \geq t_{2,i}$, 2) the inventory policy is greater than or equal to the first switchover point but less than the second switchover point $t_{1,i} \leq Q_i < t_{2,i}$ and 3) the inventory policy is less than the first switchover point, $Q_i \leq t_{1,i}$ where $i = r, v$. In the first and second cases we use the asymptotic approximation and modified approximations respectively and the renewal function and its integral are approximated by zero for the third case. In the following, we first investigate the optimal order quantity for the retailer.

Region1: When the inventory policy is greater than or equal to the second switchover point, $Q_r \geq t_{2,r}$, we plug in the asymptotic approximations of $M_G(Q_r) = \frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1$ and $\int_0^{Q_r} M_G(y)\, dy = \frac{Q_r^2}{2\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right) Q_r + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^3}$ into (3.2) to obtain

$$\frac{\partial C(Q_r)}{\partial Q_r} = \frac{m_G(Q_r)(h_r\left(Q_r + \int_0^{Q_r} M_G(y)\,dy\right) + A_r\lambda)}{(M_G(Q_r)+1)^2}$$

$$\approx \frac{h_r \frac{1}{2\mu_1^2} Q_r^2 + h_r \frac{\mu_2}{2\mu_1^3} Q_r - \frac{1}{\mu_1} A_r\lambda + h_r \frac{\mu_3}{6\mu_1^3}}{\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)^2}$$

$$= \frac{h_r}{2} - \frac{\frac{1}{\mu_1} A_r\lambda - h_r\left(\frac{\mu_3}{6\mu_1^3} - \frac{\mu_2^2}{8\mu_1^4}\right)}{\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)^2}. \tag{3.5}$$

The term of $\left(\frac{\mu_3}{6\mu_1^3} - \frac{\mu_2^2}{8\mu_1^4}\right)$ in (3.5) is nonnegative because $\frac{\mu_3}{3\mu_1} - \frac{\mu_2^2}{4\mu_1^2}$ is the variance of the excess life, γ_t, when $t \to \infty$ (Tijms, 2003). There are two cases regarding the minimum cost for the approximated $C_r(Q_r)$.

Case 1: When $h_r\left(\frac{\mu_3}{6\mu_1^2} - \frac{\mu_2^2}{8\mu_1^3}\right) \geq A_r\lambda$, the approximated $C_r(Q_r)$ increases in the region of $Q_r \geq t_{2,r}$ and yields the lowest cost at $Q_r^d = t_{2,r}$.

Case 2: When $h_r\left(\frac{\mu_3}{6\mu_1^2} - \frac{\mu_2^2}{8\mu_1^3}\right) < A_r\lambda$, the approximated $C_r(Q_r)$ is convex in the region of $Q_r \geq t_{2,r}$ and yields the lowest cost at $Q_r^d = \frac{-3h_r\mu_2 + 3\sqrt{24A_r h_r \lambda\mu_1^3 + 9h_r^2\lambda\mu_2^2 - 12\mu_1\mu_3}}{6h_r\mu_1}$ by setting $\frac{h_r}{2} - \frac{\frac{1}{1}A_r\lambda - h_r\left(\frac{{}_3}{6\mu_1^3} - \frac{{}_2^2}{8\mu_1^4}\right)}{\left(\frac{Q_r}{1} + \frac{2}{2\mu_1^2}\right)^2} = 0$ if $\frac{-3h_r\mu_2 + 3\sqrt{24A_r h_r \lambda\mu_1^3 + 9h_r^2\lambda\mu_2^2 - 12\mu_1\mu_3}}{6h_r\mu_1} \geq t_{2,r}$ or at $Q_r^d = t_{2,r}$ if $\frac{-3h_r\mu_2 + 3\sqrt{24A_r h_r \lambda\mu_1^3 + 9h_r^2\lambda\mu_2^2 - 12\mu_1\mu_3}}{6h_r\mu_1} < t_{2,r}$.

<u>Region 2</u>: When the order quantity of the retailer falls between the two switchover points, $t_{1,r} \leq Q_r < t_{2,r}$, we apply the modified approximation to (3.2) and obtain

$$\frac{\partial C_r(Q_r)}{\partial Q_r} = h_r - \frac{m_G(Q_r)\left[h_r\left(Q_r + \int_0^{Q_r} M_G(y)\,dy\right) + A_r\lambda\right]}{[M_G(Q_r)+1]^2}$$
$$= h_r - \frac{b\left[h_r\left(Q_r + a(Q_r - t_{1,r})^2\right) + A_r\lambda\right]}{\left[b(Q_r - t_{1,r}) + 1\right]^2}. \tag{3.6}$$

Here,

$$a = \frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_{2,r} + \frac{1}{2\mu_1} t_{2,r}^2}{\left(t_{2,r} - t_{1,r}\right)^2}, \tag{3.7}$$

$$b = \frac{\frac{t_{2,r}}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_{2,r} - t_{1,r}}. \tag{3.8}$$

Furthermore, we can obtain the second order derivative as

$$\frac{\partial C_r(Q_r)}{\partial Q_r^2} = -\frac{-bh_r\left(1 + 2a\left(Q_r - t_{1,r}\right)\right)\left[bQ_r - bt_{1,r} + 1\right]^2 + 2b^2\left[h_r\left(Q_r + a\left(Q_r - t_{1,r}\right)^2\right) + A_r\lambda\right]\left[bQ_r - bt_{1,r} + 1\right]}{\left[b\left(Q_r - t_{1,r}\right) + 1\right]^4}. \tag{3.9}$$

If we just have a look at Q_r that makes $\frac{\partial C_r(Q_r)}{\partial Q_r} = 0$, we have

$$\left.\frac{\partial C_r^2(Q_r)}{\partial Q_r^2}\right|_{\frac{\partial C_r(Q_r)}{\partial Q_r}=0} = -\frac{-bh_r\left(1 + 2a\left(Q_r - t_{1,r}\right)\right)\left[bQ_r - bt_{1,r} + 1\right]^2 + 2bh_r\left[bQ_r - bt_{1,r} + 1\right]^2\left[bQ_r - bt_{1,r} + 1\right]}{\left[b\left(Q_r - t_{1,r}\right) + 1\right]^4}$$
$$= \frac{bh_r\left(1 - 2(a - b)\left(Q_r - t_{1,r}\right)\right)\left[bQ_r - bt_{1,r} + 1\right]^2}{\left[b\left(Q_r - t_{1,r}\right) + 1\right]^4}. \tag{3.10}$$

Because $b > 0$, the sign of $\left.\frac{\partial C_r^2(Q_r)}{\partial Q_r^2}\right|_{\frac{\partial C(Q_r)}{\partial Q_r}=0}$ is decided by $1 - 2(a - b)\left(Q_r - t_{1,r}\right)$. When $a < b$, $\left.\frac{\partial C_r^2(Q_r)}{\partial Q_r^2}\right|_{\frac{\partial C(Q_r)}{\partial Q_r}=0} > 0$, $C(Q_r)$ is quasiconvex and unimodal. When $a > b$, we consider $1 - 2(a - b)\left(Q_r - t_{1,r}\right) > 1 - 2(a - b)\left(t_{2,r} - t_{1,r}\right)$. Furthermore, we have

$$a-b$$

$$= \frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_{2,r}+\frac{1}{2\mu_1}t_{2,r}^2-\left(t_{2,r}-t_{1,r}\right)\left(\frac{t_{2,r}}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1\right)}{\left(t_{2,r}-t_{1,r}\right)^2}$$

$$= \frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]-\frac{1}{2\mu_1}t_{2,r}^2+t_{1,r}\left(\frac{t_{2,r}}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1\right)}{\left(t_{2,r}-t_{1,r}\right)^2} \tag{3.11}$$

So,

$$1-2(a-b)\left(Q_r-t_{1,r}\right)$$

$$> 1-2\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]-\frac{1}{2\mu_1}t_{2,r}^2+t_{1,r}\left(\frac{t_{2,r}}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1\right)}{t_{2,r}-t_{1,r}}$$

$$= \frac{t_{2,r}-t_{1,r}-\left[\frac{\mu_2^2}{2\mu_1^3}-\frac{\mu_3}{3\mu_1^2}\right]+\frac{1}{\mu_1}t_{2,r}^2-t_{1,r}\left(\frac{2t_{2,r}}{\mu_1}+\frac{\mu_2}{\mu_1^2}-2\right)}{t_{2,r}-t_{1,r}}>0, \tag{3.12}$$

because $t_{2,r} \geq \mu_1$ and $t_{1,r} \leq \mu_1 - \sigma$.

In summary, $C_r(Q_r)$ is quasiconvex and unimodal in Region 2. Therefore, the minimum cost in the second region of $t_{1,r} \leq Q_r \leq t_{2,r}$ is given by $h_r\left[\left(Q_r - t_{1,r}\right) + 1\right]^2 - \left[h_r\left(Q_r + a\left(Q_r - t_{1,r}\right)^2\right) + A_r\lambda\right] = 0$ if the solution falls into the region or at the boundary point of $t_{1,r}$ and $t_{2,r}$.

Region 3: When the order quantity of the retailer is less than the first switchover points, $Q_r < t_{1,r}$, we have the approximation of both $M_G(Q_r) = 0$ and $\int_0^{Q_r} M_G(y)dy = 0$. Therefore, the minimum cost order quantity is at $Q_r = 0$.

The cost function $C_v(Q_v)$ in (3.4) is similar to (3.2) by changing $h_r, M_G(),$ and Q_r into $h_v, M_H(),$ and Q_v and considering $\frac{A_v}{(M_G(Q_r)+1)}$ equivalent to A_r. In the decentralized system, the optimal order quantity for the vendor is given in the following for a given order quantity of the retailer.

$$C_v(Q_v) = \frac{h_v(M_G(Q'^d_r) + 1)\left(Q_v + \int_{Q_r}^{Q_v}(Q_v - w)dM_H(w)\right) + A_v\lambda}{\left(M_G(Q'^d_r) + 1\right)(M_H(Q_v) + 1)}$$

$$= \frac{h_v(M_G(Q'^d_r) + 1)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v\lambda}{\left(M_G(Q'^d_r) + 1\right)(M_H(Q_v) + 1)}, \tag{3.13}$$

and

$$\frac{\partial C_v(Q_v)}{\partial Q_v} = \frac{\partial}{\partial Q_v}\left(\frac{h_v\left(M_G(Q'^d_r) + 1\right)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v\lambda}{\left(M_G(Q'^d_r) + 1\right)(M_H(Q_v) + 1)}\right)$$

$$\approx \frac{\partial}{\partial Q_v}\left(\frac{h_v\left(M_G(Q'^d_r) + 1\right)\left(Q_v + \frac{Q_v^2}{2v_1} + \left(\frac{v_2}{2v_1^2} - 1\right)Q_v + \frac{v_2^2}{4v_1^3} - \frac{v_3}{6v_1^2}\right) + A_r\lambda}{\left(M_G(Q'^d_r) + 1\right)\left(\frac{Q_v}{v_1} + \frac{v_2}{2v_1^2}\right)}\right)$$

$$= \frac{h_v\frac{1}{2v_1}\frac{1}{v_1}\left(M_G(Q'^d_r) + 1\right)Q_v^2 + 2h_v\frac{1}{2v_1}\frac{v_2}{2v_1^2}\left(M_G(Q'^d_r) + 1\right)Q_v - \frac{1}{v_1}A_v\lambda + h_v\left(M_G(Q'^d_r) + 1\right)\left(\left(\frac{v_2}{2v_1^2}\right)^2 - \frac{1}{\mu_1}\left(\frac{v_2^2}{4v_1^3} - \frac{v_3}{6v_1^2}\right)\right)}{\left(M_G(Q'^d_r) + 1\right)\left(\frac{Q_v}{v_1} + \frac{v_2}{2v_1^2}\right)^2}. \tag{3.14}$$

Since $(M_G(Q'^d_r) + 1)(\frac{Q_v}{v_1} + \frac{v_2}{2v_1^2})^2 > 0$ and $h_v\frac{1}{2v_1}\frac{1}{v_1}(M_G(Q'^d_r) + 1) > 0$ the optimal solution is given by solving

$$h_v\frac{1}{2v_1}\frac{1}{v_1}(M_G(Q_r^d) + 1)Q_v^2 + 2h_v\frac{1}{2v_1}\frac{v_2}{2v_1^2}(M_G(Q_r^d) + 1)Q_v - \frac{1}{v_1}A_v\lambda$$

$$+ h_v(M_G(Q_r^d) + 1)\left(\left(\frac{v_2}{2v_1^2}\right)^2 - \frac{1}{\mu_1}\left(\frac{v_2^2}{4v_1^3} - \frac{v_3}{6v_1^2}\right)\right) = 0 \tag{3.15}$$

which results in

$$Q_v^d = \frac{-3\vartheta_2 h_v(M_G(Q_r^d)+1) + \sqrt{3}\sqrt{24\lambda A_v \vartheta_1^3 h_v\left(M_G\left(Q_r^d\right)+1\right) + 3\vartheta_2^2 h_v\left(M_G\left(Q_r^d\right)+1\right)^2 - 4\vartheta_1\vartheta_3 h_v\left(M_G\left(Q_r^d\right)+1\right)^2}}{6\vartheta_1 h_v\left(M_G\left(Q_r^d\right)+1\right)} \tag{3.16}$$

and Q_v^d denotes the optimal inventory policy for the vendor under decentralized case using approximation of the renewal function and its integral.

We also investigate the convexity for the cost functions in the centralized system to ensure there exist a unique optimal solution in this specific case. Equation (3.17) shows per-time-unit cost function of the centralized system. Equations (3.18) and (3.19) shows the derivative of the cost function (3.17) with respect to both decision variables. For cost function (3.17) to be a convex function, the derivative functions (3.18) and (3.19) are required to be an increasing function. Similar to the decentralized system's case, since the behavior and shape of the renewal function, its integral and consequently the derivatives are not known and are different for different distribution functions, this cannot be investigated under the current forms of Equations (3.18) and (3.19).

$$\begin{aligned} C_s(Q_r, Q_v) = {} & \frac{h_r\left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + A_r\lambda}{M_G(Q_r)+1} \\ & + \frac{h_v(M_G(Q_r)+1)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v\lambda}{(M_G(Q_r)+1)(M_H(Q_v)+1)} \end{aligned} \tag{3.17}$$

$$\begin{aligned} \frac{\partial}{\partial Q_r}\left(C_s(Q_r, Q_v)\right) = {} & h_r - \frac{m_G(Q_r)\left(A_r\lambda + h_r\left(\int_0^{Q_r} M_G(y)\,dy + Q_r\right)\right)}{(M_G(Q_r)+1)^2} \\ & + \frac{m_G(Q_r)h_v\left(Q_v + \int_0^{Q_v} M_H(w)dw\right)(M_G(Q_r)+1)(M_H(Q_v)+1) - m_G(Q_r)(M_H(Q_v)+1)}{\left((M_G(Q_r)+1)(M_H(Q_v)+1)\right)^2} \end{aligned} \tag{3.18}$$

$$\frac{\partial}{\partial Q_v}\left(C_s(Q_r, Q_v)\right) = h_v - \frac{m_H(Q_v)(h(M_G(Q_r)+1)(Q_v + \int_0^{Q_v} M_H(w)dw) + A_v\lambda)}{(M_G(Q_r)+1)(M_H(Q_v)+1)^2} \tag{3.19}$$

Therefore we consider the three regions where different approximations are used and we use the earlier results of convexity analysis of the decentralized system discussed in section 3.2 since sum of two convex functions is a convex function. And the optimal solution is given by solving

$$\begin{cases} \dfrac{\partial}{\partial Q_r}\big(C_s(Q_r, Q_v)\big) = 0, \\ \dfrac{\partial}{\partial Q_v}\big(C_s(Q_r, Q_v)\big) = 0. \end{cases} \tag{3.20}$$

Approximations are used based on the value of the decision variables: 1) the decision variables are greater than or equal to the second switchover point, $Q_i \geq t_{2,i}$, 2) the decision variables are greater than or equal to first switchover point and less than the second switchover point $t_{1,i} \leq Q_i < t_{2,i}$ and 3) the decision variables are less than the first switchover point, $Q_i \leq t_{1_i}$ where $i = r, v$. In the first and second cases we use the asymptotic approximation and modified approximations respectively and the renewal function and its integral are approximated by zero for the third case. So we consider nine cases base on the values of Q_r and Q_v which may fall in either of the three regions. Due to similarity of cases we discuss the case when the inventory policy is greater than or equal to the second switchover point, $Q_r \geq t_{2,r}$ and $Q_v \geq t_{2,v}$, and we use asymptotic approximations to optimize the cost functions and to obtain the optimal inventory policies. For the other cases we may use proper approximations that are introduced in chapter 1 based on the values of Q_r and Q_v. So Equation (3.17) is approximated by

$$C_s(Q_r, Q_v) = \frac{h_r\left(\frac{1}{2\mu_1}Q_r^2 + \frac{\mu_2}{2\mu_1^2}Q_r + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right) + A_r\lambda}{\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}} + \frac{h_v\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)\left(\frac{1}{2\upsilon_1}Q_v^2 + \frac{\upsilon_2}{2\upsilon_1^2}Q_v + \frac{\upsilon_2^2}{4\upsilon_1^3} - \frac{\upsilon_3}{6\upsilon_1^2}\right) + A_v\lambda}{\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)\left(\frac{Q_v}{\upsilon_1} + \frac{\upsilon_2}{2\upsilon_1^2}\right)} \tag{3.21}$$

where

$$\vartheta_1 = \mu_1(M_G(Q_D) + 1), \tag{3.22}$$

$$\vartheta_2 = \mu_2(M_G(Q_D) + 1) + 2\mu_1(Q_D M_G(Q_D) - \int_0^{Q_D} M_G(x)\,dx) \tag{3.23}$$

and

$$\vartheta_3 = \mu_3 + 3\mu_1\int_0^{Q_D} x^2 m_G(x)dx + 3\mu_2\int_0^{Q_D} x m_G(x)dx + \mu_3\int_0^{Q_D} m_G(x)\,dx. \tag{3.24}$$

It is important to note that the moments of the distribution function $H(.)$ should be updated based on the value of Q_r. So we consider three cases as follows.

Case 1: $Q_r \geq t_{2,r}$.

$$\vartheta_1 = Q_r + \frac{\mu_2}{2\mu_1} \tag{3.25}$$

$$\vartheta_2 = {Q_r}^2 + \frac{\mu_2}{\mu_1}Q_r + \frac{\mu_3}{3\mu_1} \tag{3.26}$$

$$\vartheta_3 = {Q_r}^3 + \frac{3\mu_2}{2\mu_1}{Q_r}^2 + \frac{\mu_3}{\mu_1}Q_r + \frac{\mu_4}{4\mu_1} \tag{3.27}$$

See appendix B.1 for details.

Therefore the costs function in case of $Q_r \geq t_{2,r}$ and $Q_v \geq t_{2,v}$ would be

$C_s(Q_r, Q_v)$

$$= \frac{h_r\left(\frac{1}{2\mu_1}Q_r^2 + \frac{\mu_2}{2\mu_1^2}Q_r + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right) + A_r\lambda}{\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}}$$

$$+ \frac{h_v\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)\left(\frac{1}{2\left(Q_r + \frac{\mu_2}{2\mu_1}\right)}Q_v^2 + \frac{\left({Q_r}^2 + \frac{\mu_2}{\mu_1}Q_r + \frac{\mu_3}{3\mu_1}\right)}{2\left(Q_r + \frac{\mu_2}{2\mu_1}\right)^2}Q_v + \frac{\left({Q_r}^2 + \frac{\mu_2}{\mu_1}Q_r + \frac{\mu_3}{3\mu_1}\right)^2}{4\left(Q_r + \frac{\mu_2}{2\mu_1}\right)^3} - \frac{{Q_r}^3 + \frac{3\mu_2}{2\mu_1}{Q_r}^2 + \frac{\mu_3}{\mu_1}Q_r + \frac{\mu_4}{4\mu_1}}{6\left(Q_r + \frac{\mu_2}{2\mu_1}\right)^2}\right) + A_v\lambda}{\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2\mu_1^2}\right)\left(\frac{Q_v}{\left(Q_r + \frac{\mu_2}{2\mu_1}\right)} + \frac{\left({Q_r}^2 + \frac{\mu_2}{\mu_1}Q_r + \frac{\mu_3}{3\mu_1}\right)}{2\left(Q_r + \frac{\mu_2}{2\mu_1}\right)^2}\right)} \tag{3.28}$$

Case 2: $t_{1,r} \le Q_r < t_{2,r}$.

$$\vartheta_1 = \frac{t_2 + \frac{\mu_2}{2\mu_1} - \frac{1}{\mu_1}}{t_2 - t_1}Q_r + \mu_1\left(1 - \frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}t_1\right) \tag{3.29}$$

$$\begin{aligned}\vartheta_2 = {} & \left(2\mu_1\left(\frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1} - \frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right]t_2 + \frac{1}{2\mu_1}t_2^2}{(t_1 - t_2)^2}\right)\right)Q_r^2 \\ & + \left(\frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}\mu_2\right. \\ & \left. - 2\mu_1\left(\frac{-2t_1\left(\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right]t_2 + \frac{1}{2\mu_1}t_2^2\right)}{(t_1 - t_2)^2} + \frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}t_1\right)\right)Q_r \\ & + \left(\mu_2\left(1 - \frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}t_1\right) - 2\mu_1\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]t_1^2 + \left[\frac{\mu_2}{2\mu_1^2} - 1\right]t_1^2 t_2 + \frac{1}{2\mu_1}t_1^2 t_2^2}{(t_1 - t_2)^2}\right)\end{aligned} \tag{3.30}$$

See appendix B.2 for ϑ_3 and details.

Case 3: $Q_r < t_{1,r}$.

$$\vartheta_1 = \mu_1 \tag{3.31}$$

$$\vartheta_2 = \mu_2 \tag{3.32}$$

$$\vartheta_3 = \mu_3 \tag{3.33}$$

Now that the quasiconvexity of cost functions is ensured, we propose a method for obtaining an agreement on the decision for inventory policies of the vendor and the retailer in the decentralized case. We assume that the retailer has the market power to implement her optimal policy. For the retailer to implement any policy other her individual optimal one, the increase in her total cost must be compensated from the vendor. Suppose the vendor initiates the agreement where she offers to bear a fraction $(1-\beta)$ of the retailer's fixed cost of shipment and/or a fraction $(1-\alpha)$ of cost of inventory carried by the retailer. The costs functions are then modified as follows compared to Equations (3.1) and (3.3).

$$C_r(Q_r, \alpha, \beta) = \frac{\alpha h_r \left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + \beta A_r \lambda}{M_G(Q_r) + 1} \tag{3.34}$$

$$\begin{aligned} C_v(Q_v, \alpha, \beta) = & \frac{h_v(M_G(Q_r)+1)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v \lambda}{(M_G(Q_r)+1)(M_H(Q_v)+1)} \\ & + \frac{(1-\alpha)h_r\left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + (1-\beta)A_r\lambda}{M_G(Q_r)+1} \end{aligned} \tag{3.35}$$

The terms involving $(1-\alpha)$ and $(1-\beta)$ in Equation (3.8) reflect the cost sharing arrangement that the vendor bears.

We assume that both parties involved have complete information about their cost functions. To persuade the retailer to deviate from her individual optimal policy, the vendor makes a take-it-or-leave-it-offer to pay portion of her costs and determine retailer's order quantity. The negotiation is immediately terminated once the retailer accepts or refuses the vendor's offer while no transaction costs are assumed and the vendor has complete information about the cost functions of the retailer.

In the following two sections 3.3 and 3.4we analyze two cases of fixed cost sharing and inventory holding cost sharing for supply chain coordination.

3.3 Fixed cost sharing mechanism for supply chain coordination

3.3.1 Individual rationality

If the retailer and the vendor behave individually rational, they determine their individual optimal policies Q_r^d and Q_v^d where

$$C_r(Q_r) = \frac{h_r\left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + A_r\lambda}{M_G(Q_r) + 1}, \tag{3.36}$$

$$Q_r^d = \underset{Q_r \geq 0}{argmin}\ C(Q_r)\ and \tag{3.37}$$

$$C_v(Q_v) = \frac{h_v\left(M_G\left(Q_r^d\right) + 1\right)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v\lambda}{\left(M_G\left(Q_r^d\right) + 1\right)\left(M_H(Q_v) + 1\right)}, \tag{3.38}$$

$$Q_v^d = \underset{Q_v \geq 0}{argmin}\ C(Q_v). \tag{3.39}$$

Based on the vendor's offer, β, the retailer determines her optimal inventory policy, Q_r^* obtained from the cost function $C_r(Q_r(\beta))$ where

$$C_r(Q_r(\beta)) = \frac{h_r\left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + \beta A_r\lambda}{M_G(Q_r) + 1}, \tag{3.40}$$

and

$$Q_r^* = \underset{Q_r \geq 0}{argmin}\ C_r(Q_r(\beta)). \tag{3.41}$$

Subject to retailer's individual rationality constraint, it is required that

$$E\left[C_r\left(Q_r^d\right) - C_r(Q_r(\beta))\right] \geq 0 \tag{3.42}$$

or

$$\frac{h_r\left(Q_r^d + \int_0^{Q_r^d} M_G(y)dy\right) + A_r\lambda}{M_G\left(Q_r^d\right) + 1} - \frac{h_r\left(Q_r^* + \int_0^{Q_r^*} M_G(y)dy\right) + \beta A_r\lambda}{M_G(Q_r^*) + 1} \geq 0. \tag{3.43}$$

To ensure the retailer's individual rationality, we reduce Equation (3.43) to obtain the feasible range where contract parameter β is accepted by the retailer. This is given by Equation (3.44).

$$\beta \leq \left(\frac{h_r\left(Q_r^d + \int_0^{Q_r^d} M_G(y)dy\right) + A_r\lambda}{M_G\left(Q_r^d\right) + 1} - \frac{h_r\left(Q_r^* + \int_0^{Q_r^*} M_G(y)dy\right)}{M_G(Q_r^*) + 1}\right)\frac{M_G(Q_r^*) + 1}{A_r\lambda} \tag{3.44}$$

Also by offering β to the retailer, the vendor determines her optimal policy Q_v^* based on the cost function $C(Q_v(\beta), Q_r^*))$ where

$$C_v(Q_v(\beta), Q_r^*)) = \frac{h_v(M_G(Q_v) + 1)\left(Q_v + \int_0^{Q_v} M_H(w)dw\right) + A_v\lambda}{(M_G(Q_r^*) + 1)(M_H(Q_v^*) + 1)} + \frac{(1-\beta)A_r\lambda}{M_G(Q_r^*) + 1} \tag{3.45}$$

and

$$Q = \underset{Q_v \geq 0}{argmin}\ C_v(Q_v(\beta), Q_r^*)). \tag{3.46}$$

Subject to vendor's individual rationality constraint, it is required that

$$E[C_v(Q_v^d) - C_v(Q_v(\beta), Q_r^*))] \geq 0 \tag{3.47}$$

or

$$\begin{aligned} &\frac{h_v\left(M_G\left(Q_r^d\right) + 1\right)\left(Q_v^d + \int_0^{Q_v^d} M_H(w)dw\right) + A_v\lambda}{\left(M_G\left(Q_r^d\right) + 1\right)\left(M_H\left(Q_v^d\right) + 1\right)} \\ &\quad - \frac{h_v(M_G(Q_r^*) + 1)\left(Q_v^* + \int_0^{Q_v^*} M_H(w)dw\right) + A_v\lambda}{(M_G(Q_r^*) + 1)(M_H(Q_v^*) + 1)} - \frac{(1-\beta)A_r\lambda}{M_G(Q_r^*) + 1} \geq 0. \end{aligned} \tag{3.48}$$

To ensure the vendor's individual rationality, we reduce Equation (3.48) to obtain the feasible range of the contract parameter β for the vendor. This is given by Equation (3.49).

$$\beta \geq 1 + \left(\frac{h_v(M_G(Q_r^*) + 1)\left(Q_v^* + \int_0^{Q_v^*} M_H(w)dw\right) + A_v\lambda}{(M_G(Q_r^*) + 1)(M_H(Q_v^*) + 1)} - \frac{h_v\left(M_G\left(Q_r^d\right) + 1\right)\left(Q_v^d + \int_0^{Q_v^d} M_H(w)dw\right) + A_v\lambda}{\left(M_G\left(Q_r^d\right) + 1\right)\left(M_H\left(Q_v^d\right) + 1\right)} \right) \frac{M_G(Q_r^*) + 1}{A_r\lambda} \tag{3.49}$$

3.3.2 Optimal policies and contract

In this section we analyze that what contract parameters β are offered to retailer by the vendor. Following the preceding individual rationality analysis, if there exist a β that satisfies Equations (3.44) and (3.49), the following provides the optimal contract parameter β^*.

$$\beta^* = \underset{Q_v \geq 0, \beta}{argmin}\ C_v(Q_v(\beta), Q_r^*)) \tag{3.50}$$

In Equation (3.50), $C_v(Q_v(\beta), Q_r^*))$ is given by (3.45) and Q_r^* is a function in β given by (3.41). Using the renewal function approximation, the problem would be minimizing the following vendor's cost function with respect to β and Q_v.

$$C_v(Q_v(\beta), Q_r^*)) = \frac{h_v\left(\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)\left(\left(\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)(Q_r^* - Q_v) + \frac{{Q_v}^2}{2\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}Q_v + \frac{Q_r^{*2}}{2\vartheta_1} + \left(\frac{\vartheta_2}{2{\vartheta_1}^2} - 1\right)Q_r^*\right) + A_v\lambda}{\left(\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)\left(\frac{Q_v}{\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}\right)} + \frac{(1-\beta)A_r\lambda}{\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}}, \tag{3.51}$$

where

$$Q_r^* = \frac{-3h_r\mu_2 + \sqrt{3}\sqrt{24\beta A_r h_r \lambda {\mu_1}^3 + 3{h_r}^2{\mu_2}^2 - 4{h_r}^2\mu_1\mu_3}}{6h_r\mu_1}. \qquad (3.52)$$

If the resulting β satisfies the retailer's and the vendor's individual rationality constraints, the vendor offers the contract β to the retailer.

Proposition1. The vendor's optimal contract is achieved at $\beta = 1$ if

$$\left(M_G\left(Q^*_{r,\ }\right)+1\right)\left(M_H\left(Q^*_{v,\ }\right)+1\right) > \left(M_G\left(Q^*_{r,\ _0}\right)+1\right)\left(M_H\left(Q^*_{v,\ _0}\right)+1\right) \qquad (3.53)$$

for each β_0 where $\beta_0 < \beta$ and $Q^*_{r,}$ and $Q^*_{v,}$ represent the optimal inventory policies obtained under contract parameter β.

Proof. See appendix B.3.

3.4 Inventory holding cost sharing mechanism for supply chain coordination

3.4.1 Individual rationality

If the retailer and the vendor behave individually rational, they determine their individual optimal policies Q_r^d and Q_v^d following equations given in (3.37) and (3.39).

Based on the vendor's offer, α, the retailer determines her optimal inventory policy, Q_r^* based on the cost function $C_r(Q_r, \alpha)$ where

$$C_r(Q_r(\alpha)) = \frac{\alpha h_r\left(Q_r + \int_0^{Q_r} M_G(y)dy\right) + A_r\lambda}{M_G(Q_r)+1}, \qquad (3.54)$$

and

$$Q_r^* = \underset{Q_r \geq 0}{argmin}\ C_r(Q_r(\alpha)). \qquad (3.55)$$

Subject to retailer's individual rationality constraint, it is required that

$$E[C_r(Q_r^d) - C_r(Q_r(\alpha))] \geq 0 \qquad (3.56)$$

or

$$\frac{h_r\left(Q_r^d+\int_0^{Q_r^d}M_G(y)dy\right)+A_r\lambda}{M_G\left(Q_r^d\right)+1}-\frac{\alpha h_r\left(Q_r^*+\int_0^{Q_r^*}M_G(y)dy\right)+A_r\lambda}{M_G(Q_r^*)+1}\geq 0. \tag{3.57}$$

To ensure the retailer's individual rationality, we reduce Equation (3.57) to obtain the feasible range where contract parameter α is accepted by the retailer. This is given by Equation (3.58).

$$\alpha\leq\left(\frac{h_r\left(Q_r^d+\int_0^{Q_r^d}M_G(y)dy\right)+A_r\lambda}{M_G\left(Q_r^d\right)+1}-\frac{A_r\lambda}{M_G(Q_r^*)+1}\right)\frac{M_G(Q_r^*)+1}{h_r\left(Q_r^*+\int_0^{Q_r^*}M_G(y)dy\right)} \tag{3.58}$$

Also by offering α to the retailer, the vendor determines her optimal policy Q_v^* based on the cost function $C_v(Q_v(\alpha),Q_r^*))$ where

$$\begin{aligned}C_v(Q_v(\alpha),Q_r^*)) &= \frac{h_v(M_G(Q_r^*)+1)\left(Q_v+\int_0^{Q_v}M_H(w)dw\right)+A_v\lambda}{(M_G(Q_r^*)+1)(M_H(Q_v^*)+1)}\\ &\quad+\frac{(1-\alpha)h_r\left(Q_r^*+\int_0^{Q_r^*}M_G(y)dy\right)}{M_G(Q_r^*)+1}\end{aligned} \tag{3.59}$$

and

$$Q_{\alpha}=\underset{Q_v\geq 0}{argmin}\ C(Q_v(\alpha,Q_r^*)). \tag{3.60}$$

Subject to vendor's individual rationality constraint, it is required that

$$E[C_v(Q_v^d)-C_v(Q_v(\alpha),Q_r^*))]\geq 0 \tag{3.61}$$

or

$$\frac{h_v\left(M_G\left(Q_r^d\right)+1\right)\left(Q_v^d+\int_{Q_r^*}^{Q_v^d}\left(Q_v^d-w\right)dM_H(w)\right)+A_v\lambda}{\left(M_G\left(Q_r^d\right)+1\right)\left(M_H\left(Q_v^d\right)+1\right)} - \frac{h_v(M_G(Q_r^*)+1)\left(Q_v^*+\int_{Q_r^*}^{Q_v^*}(Q_v^*-w)dM_H(w)\right)+A_v\lambda}{(M_G(Q_r^*)+1)(M_H(Q_v^*)+1)} - \frac{(1-\alpha)h_r\left(Q_r^*+\int_0^{Q_r^*}M_G(y)dy\right)+A_r\lambda}{M_G(Q_r^*)+1} \geq 0. \tag{3.62}$$

To ensure the vendor's individual rationality, we reduce Equation (3.62) to obtain the feasible range of the contract parameter α for the vendor. This is given by Equation (3.63).

$$\alpha \geq 1 + \left(\frac{h_v\left(M_G\left(Q_r^d\right)+1\right)\left(Q_v^d+\int_{Q_r^*}^{Q_v^d}\left(Q_v^d-w\right)dM_H(w)\right)+A_v\lambda}{\left(M_G\left(Q_r^d\right)+1\right)\left(M_H\left(Q_v^d\right)+1\right)} - \frac{h_v(M_G(Q_r^*)+1)\left(Q_v^*+\int_{Q_r^*}^{Q_v^*}(Q_v^*-w)dM_H(w)\right)+A_v\lambda}{(M_G(Q_r^*)+1)(M_H(Q_v^*)+1)}\right)\frac{M_G(Q_r^*)+1}{h_r\left(Q_r^*+\int_0^{Q_r^*}M_G(y)dy\right)} \tag{3.63}$$

Therefore assuming the individual rationality of the retailer and the vendor, contract parameter α should satisfy (3.58) and (3.63).

3.4.2 Optimal policies and contract

In this section it will be analyzed that which contract parameter α are offered to the retailer by the vendor. Following the preceding individual rationality analysis, if there exist an α that satisfies Equations (3.58) and (3.63) the following provides the optimal contract parameter α^*

$$\alpha^* = \underset{Q_v \geq 0, \alpha}{argmin}\, C_v(Q_v(\alpha), Q_r^*)). \tag{3.64}$$

In Equation (3.23), $C(Q_v(\alpha), Q_r^*))$ is given by (3.64) and Q_r^* is a function in α given by (3.41). Using the approximation of the renewal function and it's integral, the problem would be minimizing the following vendor's cost function with respect to α and Q_v.

$$C_v(Q_v(\alpha), Q_r^*))$$
$$= \frac{h_v\left(\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)\left(\left(\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)(Q_r^* - Q_v) + \frac{{Q_v}^2}{2\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}Q_v + \frac{Q_r^{*2}}{2\vartheta_1} + \left(\frac{\vartheta_2}{2{\vartheta_1}^2} - 1\right)Q_r^*\right) + A_v\lambda}{\left(\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right)\left(\frac{Q_v}{\vartheta_1} + \frac{\vartheta_2}{2{\vartheta_1}^2}\right)}$$
$$+ \frac{(1-\alpha)h_r\left(\frac{Q_r^{*2}}{2\mu_1} + \frac{\mu_2}{2{\mu_1}^2}Q_r^* + \frac{{\mu_2}^2}{4{\mu_1}^3} - \frac{\mu_3}{6{\mu_1}^2}\right)}{\frac{Q_r^*}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}}, \tag{3.65}$$

where

$$Q_r^* = \frac{-3(\alpha h_r)\mu_2 + \sqrt{3}\sqrt{24A_r(\alpha h_r)\lambda{\mu_1}^3 + 3(\alpha h_r)^2{\mu_2}^2 - 4(\alpha h_r)^2\mu_1\mu_3}}{6(\alpha h_r)\mu_1}. \tag{3.66}$$

If the resulting α satisfies the retailer's and the vendor's individual rationality constraints, the vendor offer the contract α to the retailer.

3.5 Discussion on other coordination mechanisms

One approach is to consider system wide optimal contract. In section 3.3 and 3.4 we discussed the cases where the retailer imposes her optimal policy to the vendor and the vendor take the initiative and offers her optimal contract to the retailer. Assuming the individual rationality of the parties, the retailer accepts or refuses to accept the vendor's offer. However, if the parties in the system are interested in reducing the cost of the entire system by a joint decision, their problem would be to minimize system wide cost regardless of achieving the lowest individual cost.

Thus the contract parameter, α, that is given by solving

$$\alpha^* = \underset{Q_v \geq 0, \alpha}{argmin}\ C_r(Q_r(\alpha)) + C_v(Q_v(\alpha, Q_r(\alpha))) \tag{3.67}$$

minimizes the system wide cost of the supply chain. In Equation (3.67), functions $C_r(Q_r(\alpha))$, $C_v(Q_v(\alpha, Q_r(\alpha)))$ can be given by (3.34) and (3.35) respectively.

A side-payment is also often considered an effective tool to coordinate a supply chain. A side-payment is in general an additional monetary transfer between the supplier (buyer) and the buyer (supplier) that is used as an incentive for deviating from the individual optimal policy which comes under different contract types e.g. cost sharing, revenue sharing, quantity flexibility, price discount sharing, buyback, constant wholesale price, quantity discount, price discount and sales rebate. In some of these contracts the side-payment is from the buyer to the vendor e.g. the constant wholesale-price where the side-payment is the buyer's purchase quantity times the supplier's unit wholesale price and revenue sharing where the side payment is a portion of the buyer's sales revenue. In other contracts the side-payment is from the supplier to buyer including the cost sharing (Leng and Zhu 2009). In cost sharing mechanism, the side-payment that is offered to the buyer from the vendor is interpreted as a decrease in retailer's inventory holding cost. And as a result of that reduction, the buyer will be able to determine her optimal order quantity as discussed in inventory holding cost sharing mechanism earlier this chapter.

Another stream of research for supply chain coordination considers delay in payments. With permissible delay in payments, the vendor gives buyer the opportunity to invest the unpaid balance and in return expects the buyer to place larger order quantities.

3.6 Numerical experiment

We use the cost structure for the retailer and the vendor discussed in the numerical experiments section in chapter 2 to demonstrate the supply chain coordination. As an example consider the information given in Table 3.1. The retailer decides her optimal policy $Q_r^d = 73$ and realized her associated cost of $C_r(Q_r^d) = 176.92$. Without contract the vendor comes up with her optimal policy $Q_v^d = 206$ with the total cost of $C_v(Q_v^d) = 503.01$. Therefore the vendor offers the joint optimal policy and portion of the retailers holding cost, α to minimize her cost. Figure 3.1 shows the cost savings for the information given in Table 3.1. By implementing the joint inventory policy and deviating from her individual optimal policy, the retailer's total cost decreases to $C_r(Q_r^*) = 66.83$. Also, the vendor's total cost decreases to $C_v(Q_v^*) = 472.69$.

Table 3.1 Sample supply chain cost and inventory decisions information

$\mu_1 = 20$	$\lambda = 0.1$		
$A_r = 40$	$h_r = 2$	$Q_r^d = 73$	$Q_r(\alpha) = 224$
$A_v = 40$	$h_v = 4$	$Q_v^d = 206$	$Q_v(\alpha) = 0$
$C_r(Q_r^d) + C_v(Q_v^d) = 679.93$			
$C_s(Q_r(\alpha), Q_v(\alpha)) = 539.52, \alpha = 14\%$			
$C_s(Q_r^*, Q_v^*) = 536.01$			

Under the complete information assumption, $\alpha = 14\%$ and the system wide cost saving of coordination is 20.6% while the centralization results in 21.2% cost saving. As discussed in section 2.5, the significant saving are expected to be observed where there is a shift in inventory resulting in zero order quantity of the vendor. This can be also seen in this numerical example where the contract is offered at $Q_v(\alpha) = 0$.

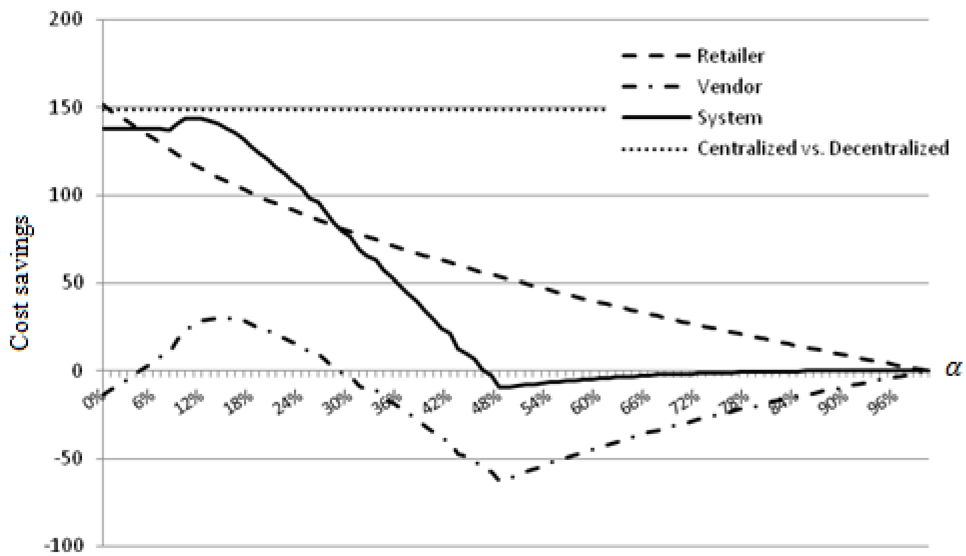

Figure 3.1 Cost savings of the coordinated supply chain (information in Table 3.1)

3.7 Conclusion

In this study we consider a two-echelon supply chain system discussed in Chapter two and investigate the convexity of centralized and decentralized systems' cost functions considering the proposed approximation in chapter one to ensure there exist a unique optimal solution for the system regarding the cost and inventory policy decisions. It is well known that implementing the joint optimal policies always results in savings in total cost of the system. However, usually one party has more power over the other to impose her individual optimal policy and ask the other party for incentives in order to act cooperatively. As a coordination tool, we proposed a cost sharing mechanism with which the vendor offers the retailer a contract as a compensation of implementing vendor-desired inventory policy. Assuming the complete information model, we discussed about sharing of inventory holding cost and fixed cost. The solution to the models results in a contract offered by the vendor which aims to minimize the vendors cost while satisfying the individual rationality of the retailer. We also discussed about some other form of coordination mechanisms including side payment, delayed payment and the contract that provide system wide optimal decisions and cost.

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APPENDIX A

ADDITIONAL DETAILS FOR CHAPTER ONE

A.1 Comparison of Approximations for Weibull Distributions

Table A.1 Comparison of approximations for Weibull distributions

Distribution	*Mean*	*Variance*	*Deviation from simulation results*			
			$M(t)$		$\int_0^t M(x)\,dx$	
			Asy. App.[1]	*Mod. App.*[2]	*Asy. App.*	*Mod. App.*
Weibull (5, 20)	4.86	0.91	0.107	0.025	0.236	0.228
Weibull (10, 20)	9.73	0.36	0.148	0.125	0.290	0.260
Weibull (20, 20)	19.47	1.45	0.179	0.136	0.529	0.416
Weibull (40, 20)	38.94	5.82	0.200	0.116	1.167	0.734
Weibull (2, 5)	1.8	0.17	0.030	0.025	1.074	1.072
Weibull (5, 5)	4.59	1.11	0.016	0.008	0.241	0.235
Weibull (10, 5)	9.18	4.42	0.026	0.013	0.178	0.160
Weibull (20, 5)	18.36	17.69	0.048	0.024	0.193	0.129
Weibull (1, 2)	0.886	0.21	0.039	0.036	2.263	2.262
Weibull (10, 2)	8.86	21.46	0.012	0.005	0.184	0.168
Weibull (20, 2)	17.72	85.84	0.020	0.008	0.155	0.102
Weibull (40, 2)	35.44	343.36	0.038	0.015	0.315	0.116
Weibull (1, 1.5)	0.902	0.38	0.088	0.085	3.584	3.583
Weibull (5, 1.5)	4.51	9.39	0.013	0.009	0.394	0.388
Weibull (15, 1.5)	13.54	84.53	0.014	0.004	0.169	0.130
Weibull (30, 1.5)	27.08	338.12	0.023	0.005	0.245	0.099
Weibull (0.1, 0.5)	0.2	0.2	0.570	0.551	31.641	31.630
Weibull (0.5, 0.5)	1	5	0.156	0.125	3.806	3.692
Weibull (1, 0.5)	2	20	0.156	0.109	4.212	3.823
Weibull (5, 0.5)	10	500	0.485	0.304	27.596	19.117

1) Asymptotic approximation,
2) Modified approximation

A.2 Comparison of approximation for different methods of obtaining switch-over points

Table A.2 Average deviation of approximated $M(t)$ from simulation result using method I

Distribution	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	Mod. App.	Asy. App.	Mod. App.
Weibull (5, 20)	0.2271	0.0150	0.2556	0.0881
Weibull (10, 20)	0.2248	0.0150	0.2547	0.0872
Weibull (20, 20)	0.2333	0.0152	0.2743	0.0964
Weibull (40, 20)	0.2264	0.0149	0.2563	0.0818
Weibull (2, 5)	0.1627	0.0425	0.0934	0.0815
Weibull (5, 5)	0.1634	0.0430	0.1089	0.0998
Weibull (10, 5)	0.1632	0.0429	0.0924	0.0811
Weibull (20, 5)	0.1632	0.0429	0.0942	0.0824
Weibull (1, 2)	0.1017	0.0380	0.0605	0.0448
Weibull (10, 2)	0.1018	0.0384	0.0607	0.0453
Weibull (20, 2)	0.1019	0.0378	0.0607	0.0445
Weibull (40, 2)	0.1029	0.0376	0.0615	0.0444
Weibull (1, 1.5)	0.0925	0.0187	0.0761	0.0199
Weibull (5, 1.5)	0.0920	0.0191	0.0755	0.0204
Weibull (15, 1.5)	0.0931	0.0188	0.0765	0.0201
Weibull (30, 1.5)	0.0935	0.0192	0.0761	0.0206
Weibull (0.1, 0.5)	0.9523	0.2769	0.9120	0.2876
Weibull (0.5, 0.5)	0.9552	0.2793	0.9150	0.2900
Weibull (1, 0.5)	0.8833	0.3184	0.8473	0.3287
Weibull (5, 0.5)	0.9567	0.2784	0.9165	0.2891

Table A.3 Average deviation of approximated $M(t)$ from simulation result using method II

Distribution	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	Mod. App.	Asy. App.	Mod. App.
Weibull (5, 20)	0.2286	0.0192	0.1536	0.2645
Weibull (10, 20)	0.2269	0.0154	0.1500	0.1031
Weibull (20, 20)	0.2333	0.0102	0.2437	0.0755
Weibull (40, 20)	0.2309	0.0147	0.2833	0.1055
Weibull (2, 5)	0.1661	0.0668	0.1047	0.0561
Weibull (5, 5)	0.1659	0.0323	0.1076	0.0581
Weibull (10, 5)	0.1658	0.0347	0.1073	0.0575
Weibull (20, 5)	0.1665	0.0344	0.1064	0.0557
Weibull (1, 2)	0.0967	0.0617	0.0151	0.0650
Weibull (10, 2)	0.0969	0.0611	0.0159	0.0648
Weibull (20, 2)	0.0988	0.0629	0.0153	0.0679
Weibull (40, 2)	0.0977	0.0619	0.0160	0.0675
Weibull (1, 1.5)	0.0816	0.0628	0.0378	0.0632
Weibull (5, 1.5)	0.0811	0.0622	0.0381	0.0640
Weibull (15, 1.5)	0.0818	0.0624	0.0382	0.0642
Weibull (30, 1.5)	0.0811	0.0614	0.0367	0.0622
Weibull (0.1, 0.5)	1.3082	0.2913	1.3082	0.2913
Weibull (0.5, 0.5)	1.3087	0.2908	1.3087	0.2908
Weibull (1, 0.5)	1.3052	0.2898	1.3052	0.2898
Weibull (5, 0.5)	1.3100	0.2923	1.3100	0.2923

Table A.4 Average deviation of approximated $M(t)$ from simulation result using method III

Distribution	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	Mod. App.	Asy. App.	Mod. App.
Weibull (5, 20)	0.1604	0.1762	0.1460	0.2010
Weibull (10, 20)	0.1591	0.1802	0.1493	0.2028
Weibull (20, 20)	0.1593	0.1826	0.1512	0.2045
Weibull (40, 20)	0.1582	0.1831	0.1510	0.2044
Weibull (2, 5)	0.1013	0.0445	0.0613	0.0571
Weibull (5, 5)	0.1011	0.0446	0.0617	0.0591
Weibull (10, 5)	0.1012	0.0448	0.0611	0.0571
Weibull (20, 5)	0.1017	0.0449	0.0618	0.0577
Weibull (1, 2)	0.0967	0.0426	0.0572	0.0499
Weibull (10, 2)	0.969	0.0431	0.0575	0.0505
Weibull (20, 2)	0.9880	0.0431	0.0587	0.0506
Weibull (40, 2)	0.0977	0.0422	0.0581	0.0494
Weibull (1, 1.5)	0.0816	0.0265	0.0665	0.0283
Weibull (5, 1.5)	0.0811	0.0269	0.0659	0.0288
Weibull (15, 1.5)	0.0818	0.0268	0.0666	0.0287
Weibull (30, 1.5)	0.0811	0.0270	0.0653	0.0290
Weibull (0.1, 0.5)	0.6622	0.5779	0.6388	0.5881
Weibull (0.5, 0.5)	0.5785	0.4621	0.5593	0.4683
Weibull (1, 0.5)	0.5641	0.4689	0.5446	0.4753
Weibull (5, 0.5)	0.5815	0.4614	0.5624	0.4676

Table A.5 Average deviation of approximated $\int_0^t M(x)dx$ from simulation result using method I

Distribution	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	Mod. App.	Asy. App.	Mod. App.
Weibull (5, 20)	0.1518	0.0156	0.2203	0.1133
Weibull (10, 20)	0.3060	0.0363	0.4366	0.2229
Weibull (20, 20)	0.6275	0.0581	0.8526	0.3841
Weibull (40, 20)	1.2414	0.1483	1.6687	0.8396
Weibull (2, 5)	0.0414	0.0096	0.0279	0.0187
Weibull (5, 5)	0.1030	0.0250	0.0525	0.0655
Weibull (10, 5)	0.2061	0.0490	0.1440	0.0941
Weibull (20, 5)	0.4164	0.1020	0.2852	0.1996
Weibull (1, 2)	0.0164	0.0024	0.0080	0.0029
Weibull (10, 2)	0.1635	0.0247	0.0790	0.0296
Weibull (20, 2)	0.3282	0.0495	0.1595	0.0592
Weibull (40, 2)	0.6691	0.0919	0.3301	0.1099
Weibull (1, 1.5)	0.0219	0.0024	0.0172	0.0026
Weibull (5, 1.5)	0.1111	0.0133	0.0875	0.0148
Weibull (15, 1.5)	0.3379	0.0408	0.2666	0.0451
Weibull (30, 1.5)	0.6837	0.0827	0.5348	0.0920
Weibull (0.1, 0.5)	0.9165	0.2821	0.9055	0.2929
Weibull (0.5, 0.5)	4.5806	1.4088	4.5260	1.4630
Weibull (1, 0.5)	8.8064	3.4540	8.7034	3.5655
Weibull (5, 0.5)	45.7896	14.0718	45.2431	14.6130

Table A.6 Average deviation of approximated $\int_0^t M(x)dx$ from simulation result using method II

Distribution	$[0, t_2)$		$[t_1, t_2)$	
	Asy. App.	Mod. App.	Asy. App.	Mod. App.
Weibull (5, 20)	0.1504	0.0040	0.2901	0.0655
Weibull (10, 20)	0.3030	0.0123	0.5725	0.1883
Weibull (20, 20)	0.6275	0.0299	1.0057	0.2794
Weibull (40, 20)	1.2125	0.0972	1.7112	0.7408
Weibull (2, 5)	0.0415	0.0198	0.0162	0.0323
Weibull (5, 5)	0.1033	0.0263	0.0440	0.0770
Weibull (10, 5)	0.2068	0.0571	0.0839	0.1555
Weibull (20, 5)	0.4176	0.1121	0.1799	0.3060
Weibull (1, 2)	0.0157	0.0180	0.0040	0.0291
Weibull (10, 2)	0.1560	0.1791	0.0383	0.2891
Weibull (20, 2)	0.3197	0.3668	0.0821	0.6042
Weibull (40, 2)	0.6400	0.7281	0.1711	1.1817
Weibull (1, 1.5)	0.0191	0.0190	0.0067	0.0258
Weibull (5, 1.5)	0.0970	0.0934	0.0374	0.1259
Weibull (15, 1.5)	0.2944	0.2809	0.1124	0.3807
Weibull (30, 1.5)	0.5893	0.5526	0.2225	0.7483
Weibull (0.1, 0.5)	1.0750	0.2830	1.0750	0.2830
Weibull (0.5, 0.5)	5.3756	1.4154	5.3756	1.4154
Weibull (1, 0.5)	10.7537	2.8337	10.7537	2.8337
Weibull (5, 0.5)	53.7561	14.1561	53.7561	14.1561

Table A.7 Average deviation of approximated $\int_0^t M(x)dx$ from simulation result using method III

Distribution	$[0, t_2)$ Asy. App.	$[0, t_2)$ Mod. App.	$[t_1, t_2)$ Asy. App.	$[t_1, t_2)$ Mod. App.
Weibull (5, 20)	1.8497	0.1139	2.1139	1.8497
Weibull (10, 20)	3.7244	0.2694	4.1965	3.7244
Weibull (20, 20)	7.4963	0.4904	8.4001	7.4963
Weibull (40, 20)	15.0670	1.0199	16.8300	15.0670
Weibull (2, 5)	0.0166	0.0192	0.0215	0.0166
Weibull (5, 5)	0.0452	0.0443	0.0634	0.0452
Weibull (10, 5)	0.0836	0.1037	0.1154	0.0836
Weibull (20, 5)	0.1874	0.2169	0.2435	0.1874
Weibull (1, 2)	0.0037	0.0077	0.0040	0.0037
Weibull (10, 2)	0.0346	0.0757	0.0411	0.0346
Weibull (20, 2)	0.0681	0.1562	0.0813	0.0681
Weibull (40, 2)	0.1301	0.3180	0.1547	0.1301
Weibull (1, 1.5)	0.0035	0.0147	0.0038	0.0035
Weibull (5, 1.5)	0.0192	0.0755	0.0209	0.0192
Weibull (15, 1.5)	0.0576	0.2292	0.0630	0.0576
Weibull (30, 1.5)	0.1173	0.4547	0.1285	0.1173
Weibull (0.1, 0.5)	0.3186	0.7199	0.3242	0.3186
Weibull (0.5, 0.5)	1.7050	3.2077	1.7281	1.7050
Weibull (1, 0.5)	3.5252	6.5322	3.5728	3.5252
Weibull (5, 0.5)	17.1609	32.1893	17.3928	17.1609

A.3 Details of obtaining the supply chain inventory holding cost

The waiting penalty cost in Equation (1.19) can be computed based on $\int_0^{Q_D} y dM_G(y) = Q_D M_G(Q_D) - \int_0^{Q_D} M_G(y)dy$, where the integral of renewal function has a negative sign. If the asymptotic approximation is used to minimize the cost function, decreasing the values of Q_D and consequently increasing the value of $\int_0^{Q_D} M_G(y)dy$ will reduce the waiting penalty cost and the total cost. However, if the modified approximation is used, $\int_0^{Q_D} M_G(y)dy$ does not increase at small values of Q_D.

The inventory holding cost in Equation (1.19) is calculated based on

$$\begin{aligned}\int_{Q_D}^{Q_R}(Q_R - w)dM_H(w) &= Q_R \int_{Q_D}^{Q_R} dM_H(w) - \int_{Q_D}^{Q_R} w dM_H(w) \\ &= M_H(Q_D)(Q_D - Q_R) + \left(\int_0^{Q_R} M_H(w)dw - \int_0^{Q_D} M_H(w)dw\right).\end{aligned} \tag{A.1}$$

If the modified approximation is used, since $Q_R > Q_D$, the renewal function and its integral of excess life at the age of initial renewal process, $M_H(Q_D)$ and $\int_0^{Q_D} M_H(w)dw$, would get zero values and the equation for calculating the inventory holding cost reduces to $\int_{Q_D}^{Q_R}(Q_R - w)dM_H(w) = \int_0^{Q_R} M_H(w)dw$.

A.4 Proof of Theorem 1

The asymptotic result $\int_0^t M(x)dx \to \frac{t^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)t + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]$ as $t \to \infty$ suggests that for some constant c,

$$\int_0^s \int_0^t M(x)dx\, dt \approx \int_0^s \left(\left[\frac{t^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)t\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]\right)dt + c \tag{A.2}$$

for large t. To determine the constant c, we define the function

$$Z(s) = \int_0^s \int_0^t M(x)dx\, dt - \left[\frac{s^3}{6\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)\frac{s^2}{2}\right] - \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]s \quad t, s \geq 0 \tag{A.3}$$

By integrating both sides and interchanging the order of integration, we get the following renewal equation for the function

$$\int_0^s \int_0^{s-y} \int_0^t M(x)dx\,dtf(y)dy$$

$$= \int_0^s Z(s-y)f(y)dy$$

$$+ \int_0^s \left(\left[\frac{(s-y)^3}{6\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1 \right) \frac{(s-y)^2}{2} \right] \right. \tag{A.4}$$

$$\left. + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} \right] (s-y) \right) f(y)dy$$

From this renewal equation, we can obtain, $Z(s) = a(s) + \int_0^s Z(s-y)\,f(y)dy$

where

$$a(s) = \int_0^s \int_0^t F(x)dx\,dt - \left[\frac{s^3}{6\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1 \right) \frac{s^2}{2} \right] - \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} \right] s$$

$$+ \int_0^s \left(\left[\frac{(s-y)^3}{6\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1 \right) \frac{(s-y)^2}{2} \right] \right. \tag{A.5}$$

$$\left. + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} \right] (s-y) \right) f(y)dy.$$

Now by using the key renewal theorem we can get

$$\lim_{s \to \infty} Z(s) = \frac{1}{\mu_1} \int_0^\infty a(s)ds = \frac{\mu_4}{24\mu_1^2} - \frac{\mu_3\mu_2}{6\mu_1^3} + \frac{\mu_2^3}{8\mu_1^4}. \tag{A.6}$$

So the constant term $c = \frac{\mu_4}{24\mu_1^2} - \frac{\mu_3\mu_2}{6\mu_1^3} + \frac{\mu_2^3}{8\mu_1^4}$ and we have

$$\lim_{t\to\infty}\left[\int_0^t\int_0^s M(x)dx\,ds-\left\{\left[\frac{t^3}{6\mu_1}+\left(\frac{\mu_2}{2{\mu_1}^2}-1\right)\frac{t^2}{2}\right]+\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]t\right\}\right]$$
$$=\frac{\mu_4}{24\mu_1^2}-\frac{\mu_2\mu_3}{6\mu_1^3}+\frac{\mu_2^3}{8\mu_1^4}. \tag{A.7}$$

■

APPENDIX B

ADITIONAL DETAILES FOR CHAPTER THREE

B.1 Renewal function moments for the region 1 approximation

$$v_1 = \mu_1\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right) = Q_r + \frac{\mu_2}{2\mu_1} \tag{B.1}$$

$$\begin{aligned} v_2 &= \mu_2(M_G(Q_r) + 1) + 2\mu_1\left(Q_r M_G(Q_r) - \int_0^{Q_r} M_G(y)dy\right) \\ &= \mu_2\left(\frac{Q_r}{\mu_1} + \frac{\mu_2}{2{\mu_1}^2}\right) + 2\mu_1\left(\frac{{Q_r}^2}{2\mu_1} - \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]\right) \\ &= {Q_r}^2 + \frac{\mu_2 Q_r}{\mu_1} + \frac{\mu_2^2}{2{\mu_1}^2} - \frac{\mu_2^2}{2{\mu_1}^2} + \frac{\mu_3}{3\mu_1} = {Q_r}^2 + \frac{\mu_2}{\mu_1}Q_r + \frac{\mu_3}{3\mu_1}, \text{ and} \end{aligned} \tag{B.2}$$

$$v_3 = \mu_3 + 3\mu_1\int_0^{Q_r} y^2 m_G(y)dy + 3\mu_2\int_0^{Q_r} y m_G(y)dy + \mu_3\int_0^{Q_r} m_G(y)dy, \tag{B.3}$$

where μ_1, μ_2 and μ_3 denote the first, second and third moment of the order size Y_n and

$$\int_0^{Q_r} y^2 m_G(y)dy = {Q_r}^2 M_G(Q_r) - 2\left[Q_r\int_0^{Q_r} M_G(y)dy - \int_0^{Q_r}\int_0^{y} M_G(x)dx\, dy\right] \tag{B.4}$$

where

$$\begin{aligned} &\int_0^{Q_r}\int_0^{y} M_G(x)dx\, dy \\ &\quad\approx \left[\frac{{Q_r}^3}{6\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)\frac{{Q_r}^2}{2}\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]Q_r + \frac{\mu_4}{24{\mu_1}^2} - \frac{\mu_3\mu_2}{6{\mu_1}^3} + \frac{\mu_2^3}{8\mu_1^4}. \end{aligned} \tag{B.5}$$

Since

$$\int_0^{Q_r} m_G(y)dy = M_G(Q_r) = \frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}, \tag{B.6}$$

$$\begin{aligned} \int_0^{Q_r} y m_G(y)dy &= Q_r M_G(Q_r) - \int_0^{Q_r} M_G(y)dy \\ &= Q_r\left(\frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}\right) - \left(\left[\frac{{Q_r}^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)Q_r\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]\right), \end{aligned} \tag{B.7}$$

and

$$\int_0^{Q_r} y^2 m_G(y)dy = {Q_r}^2 M_G(Q_r) - 2\left[Q_r \int_0^{Q_r} M_G(y)dy - \int_0^{Q_r}\int_0^{y} M_G(x)dx\,dy\right]$$
$$= {Q_r}^2\left(\frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}\right)$$
$$- 2\left[Q_r\left(\left[\frac{{Q_r}^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)Q_r\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]\right)\right.$$
$$- \left(\left[\frac{{Q_r}^3}{6\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)\frac{{Q_r}^2}{2}\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]Q_r + \frac{\mu_4}{24{\mu_1}^2} - \frac{\mu_3\mu_2}{6{\mu_1}^3} \tag{B.8}$$
$$\left.\left. + \frac{\mu_2^3}{8\mu_1^4}\right)\right]$$

So we have

$$\upsilon_3 = \mu_3$$
$$+ 3\mu_1\left({Q_r}^2\left(\frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}\right)\right.$$
$$- 2\left[Q_r\left(\left[\frac{{Q_r}^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)Q_r\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]\right)\right.$$
$$\left.\left. - \left(\left[\frac{{Q_r}^3}{6\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)\frac{{Q_r}^2}{2}\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]Q_r + \frac{\mu_4}{24{\mu_1}^2} - \frac{\mu_3\mu_2}{6{\mu_1}^3} + \frac{\mu_2^3}{8\mu_1^4}\right)\right]\right)$$
$$+ 3\mu_2\left(Q_r\left(\frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}\right) - \left(\left[\frac{{Q_r}^2}{2\mu_1} + \left(\frac{\mu_2}{2{\mu_1}^2} - 1\right)Q_r\right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]\right)\right) \tag{B.9}$$
$$+ \mu_3\left(\frac{Q_r}{\mu_1} + \frac{\mu_2 - 2{\mu_1}^2}{2{\mu_1}^2}\right) = {Q_r}^3 + \frac{3\mu_2}{2\mu_1}{Q_r}^2 + \frac{\mu_3}{\mu_1}Q_r + \frac{\mu_4}{4\mu_1}$$

B.2 Renewal function moments for the region 2 approximation

$$\upsilon_1 = \mu_1(M_G(Q_r)+1) = \mu_1\left(\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}(Q_r-t_1)+1\right)$$

$$= \frac{t_2+\frac{\mu_2}{2\mu_1}-\frac{1}{\mu_1}}{t_2-t_1}Q_r+\mu_1\left(1-\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}t_1\right), \tag{B.10}$$

$$\upsilon_2 = \mu_2(M_G(Q_r)+1)+2\mu_1\left(Q_rM_G(Q_r)-\int_0^{Q_r}M_G(y)dy\right)$$

$$= \mu_2\left(\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}(Q_r-t_1)+1\right)$$

$$+2\mu_1\left(Q_r\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}(Q_r-t_1)\right.$$

$$-\left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_2+\frac{1}{2\mu_1}t_2^2}{(t_1-t_2)^2}\right)Q_r^2 \tag{B.11}$$

$$+\left(\frac{-2t_1\left(\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_2+\frac{1}{2\mu_1}t_2^2\right)}{(t_1-t_2)^2}\right)Q_r$$

$$\left.+\left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]t_1^2+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_1^2t_2+\frac{1}{2\mu_1}t_1^2t_2^2}{(t_1-t_2)^2}\right)\right), \text{ or}$$

$$
\begin{aligned}
\upsilon_2 &= \mu_2(M_G(Q_r)+1) + 2\mu_1\left(Q_r M_G(Q_r) - \int_0^{Q_r} M_G(y)dy\right) \\
&= \mu_2\left(\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}(Q_r-t_1)+1\right) \\
&\qquad + 2\mu_1\left(Q_r\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}(Q_r-t_1)\right. \\
&\qquad - \left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_2+\frac{1}{2\mu_1}t_2^2}{(t_1-t_2)^2}\right)Q_r^2 \\
&\qquad + \left(\frac{-2t_1\left(\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_2+\frac{1}{2\mu_1}t_2^2\right)}{(t_1-t_2)^2}\right)Q_r \\
&\qquad \left. + \left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]t_1^2+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_1^2 t_2+\frac{1}{2\mu_1}t_1^2 t_2^2}{(t_1-t_2)^2}\right)\right), \text{and}
\end{aligned}
\tag{B.12}
$$

$$
\upsilon_3 = \mu_3 + 3\mu_1\int_0^{Q_r} y^2 m_G(y)dy + 3\mu_2\int_0^{Q_r} y m_G(y)dy + \mu_3\int_0^{Q_r} m_G(y)dy, \tag{B.13}
$$

where μ_1, μ_2 and μ_3 denote the first, second and third moment of the order size Y_n and

$$
\int_0^{Q_r} y^2 m_G(y)dy = {Q_r}^2 M_G(Q_r) - 2\left[Q_r\int_0^{Q_r} M_G(y)dy - \int_0^{Q_r}\int_0^{y} M_G(x)dx\,dy\right], \tag{B.14}
$$

where

$$\int_0^{Q_r} y^2 m_G(y)dy = {Q_r}^2 M_G(Q_r) - 2\left[Q_r \int_0^{Q_r} M_G(y)dy - \int_0^{Q_r}\int_0^{y} M_G(x)dx\,dy\right] \tag{B.15}$$

So to obtain υ_3 we have

$$\int_0^{Q_r} m_G(y)dy = M_G(Q_r) = \frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}(Q_r - t_1), \text{and} \tag{B.16}$$

$$\begin{aligned}\int_0^{Q_r} y m_G(y)dy &= Q_r M_G(Q_r) - \int_0^{Q_r} M_G(y)dy \\ &= Q_r\left(\frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1}(Q_r - t_1)\right) \\ &\quad - \left(\left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right]t_2 + \frac{1}{2\mu_1}t_2^2}{(t_1 - t_2)^2}\right)Q_r^2\right. \\ &\quad + \left(\frac{-2t_1\left(\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right]t_2 + \frac{1}{2\mu_1}t_2^2\right)}{(t_1 - t_2)^2}\right)Q_r \\ &\quad \left. + \left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right]t_1^2 + \left[\frac{\mu_2}{2\mu_1^2} - 1\right]t_1^2 t_2 + \frac{1}{2\mu_1}t_1^2 t_2^2}{(t_1 - t_2)^2}\right)\right),\end{aligned} \tag{B.117}$$

and

$$\int_0^{Q_r} y^2 m_G(y)dy = {Q_r}^2 M_G(Q_r) - 2\left[Q_r \int_0^{Q_r} M_G(y)dy - \int_0^{Q_r}\int_0^{y} M_G(x)dx\,dy\right]$$

$$= {Q_r}^2\left(\frac{\frac{t_2}{\mu_1}+\frac{\mu_2}{2\mu_1^2}-1}{t_2-t_1}(Q_r-t_1)\right)$$

$$-2\left[Q_r\left(\left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_2+\frac{1}{2\mu_1}t_2^2}{(t_1-t_2)^2}\right)Q_r^2\right.\right.$$

$$+\left(\frac{-2t_1\left(\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_2+\frac{1}{2\mu_1}t_2^2\right)}{(t_1-t_2)^2}\right)Q_r$$

$$\left.+\left(\frac{\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]t_1^2+\left[\frac{\mu_2}{2\mu_1^2}-1\right]t_1^2t_2+\frac{1}{2\mu_1}t_1^2t_2^2}{(t_1-t_2)^2}\right)\right)$$

$$-\left(\left[\frac{{Q_r}^3}{6\mu_1}+\left(\frac{\mu_2}{2{\mu_1}^2}-1\right)\frac{{Q_r}^2}{2}\right]+\left[\frac{\mu_2^2}{4\mu_1^3}-\frac{\mu_3}{6\mu_1^2}\right]Q_r+\frac{\mu_4}{24{\mu_1}^2}-\frac{\mu_3\mu_2}{6{\mu_1}^3}\right.$$

$$\left.\left.+\frac{\mu_2^3}{8\mu_1^4}\right)\right]. \tag{B.18}$$

Therefore

$$
\begin{aligned}
v_3 &= \mu_3 + 3\mu_1 \int_0^{Q_r} y^2 m_G(y)dy + 3\mu_2 \int_0^{Q_r} y m_G(y)dy + \mu_3 \int_0^{Q_r} m_G(y)dy \\
&= \mu_3 \\
&\quad + 3\mu_1 \Bigg(Q_r{}^2 \Bigg(\frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1} (Q_r - t_1) \Bigg) \\
&\quad - 2 \Bigg[Q_r \Bigg(\Bigg(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_2 + \frac{1}{2\mu_1} t_2^2}{(t_1 - t_2)^2} \Bigg) Q_r^2 \\
&\quad + \Bigg(\frac{-2t_1 \left(\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_2 + \frac{1}{2\mu_1} t_2^2 \right)}{(t_1 - t_2)^2} \Bigg) Q_r \\
&\quad + \Bigg(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] t_1^2 + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_1^2 t_2 + \frac{1}{2\mu_1} t_1^2 t_2^2}{(t_1 - t_2)^2} \Bigg) \Bigg) \\
&\quad - \Bigg(\left[\frac{Q_r{}^3}{6\mu_1} + \left(\frac{\mu_2}{2\mu_1{}^2} - 1 \right) \frac{Q_r{}^2}{2} \right] + \left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} \right] Q_r + \frac{\mu_4}{24\mu_1{}^2} - \frac{\mu_3\mu_2}{6\mu_1{}^3} + \frac{\mu_2^3}{8\mu_1^4} \Bigg) \Bigg] \Bigg) \\
&\quad + 3\mu_2 \Bigg(Q_r \Bigg(\frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1} (Q_r - t_1) \Bigg) \\
&\quad - \Bigg(\Bigg(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_2 + \frac{1}{2\mu_1} t_2^2}{(t_1 - t_2)^2} \Bigg) Q_r^2 \\
&\quad + \Bigg(\frac{-2t_1 \left(\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_2 + \frac{1}{2\mu_1} t_2^2 \right)}{(t_1 - t_2)^2} \Bigg) Q_r \\
&\quad + \Bigg(\frac{\left[\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right] t_1^2 + \left[\frac{\mu_2}{2\mu_1^2} - 1\right] t_1^2 t_2 + \frac{1}{2\mu_1} t_1^2 t_2^2}{(t_1 - t_2)^2} \Bigg) \Bigg) \Bigg) + \mu_3 \Bigg(\frac{\frac{t_2}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1}{t_2 - t_1} (Q_r - t_1) \Bigg).
\end{aligned}
\tag{B.19}
$$

B.3 Proof of Proposition 1

Consider the individual rationality constraint of the vendor.

$$\frac{h_v\left(M_G\left(Q_r^d\right)+1\right)\left(Q_v^d+\int_0^{Q_v^d} M_H(w)dw\right)+A_v\lambda}{\left(M_G\left(Q_r^d\right)+1\right)\left(M_H\left(Q_v^d\right)+1\right)} \\ -\frac{h_v\left(Q_{v,\ }^*+\int_0^{Q_{v,\beta}^*} M_H(w)dw\right)+A_v\lambda}{\left(M_H\left(Q_{v,\ }^*\right)+1\right)}\frac{(1-\beta)A_r\lambda}{M_G\left(Q_{r,\ }^*\right)+1}\geq 0 \tag{B.20}$$

Suppose a contract parameter β_0 where $\beta_0 < \beta$ and that $Q_{r,\ }^*$ and $Q_{v,\ }^*$ are the optimal inventory policies obtained under contract parameter β. We show the condition under which by decreasing the contract parameter β to β_0 the vendors cost increases i.e. the above constraint holds as equality only if $\beta = 1$. Since $\beta_0 < \beta$ we have that $Q_{r,\ 0}^* < Q_{r,\ }^*$ and consequently $M_G\left(Q_{r,\ 0}^*\right) < M_G\left(Q_{r,\ }^*\right)$ or $Q_{v,\ 0}^* > Q_{v,\ }^*$ which also results in $M_H\left(Q_{v,\ 0}^*\right) > M_H\left(Q_{v,\ }^*\right)$ and $\int_0^{Q_{v,\beta_0}^*} M_H(w)dw > \int_0^{Q_{v,\beta}^*} M_H(w)dw$. So we have $(1-\beta_0)A_r\lambda > (1-\beta)A_r\lambda$ and

$$h_v\left(Q_{v,\ 0}^*+\int_0^{Q_{v,\beta_0}^*} M_H(w)dw\right)+A_v\lambda > h_v\left(Q_{v,\ }^*+\int_0^{Q_{v,\beta}^*} M_H(w)dw\right)+A_v\lambda. \tag{B.21}$$

Thus if $\left(M_G\left(Q_{r,\ }^*\right)+1\right)\left(M_H\left(Q_{v,\ }^*\right)+1\right)$ is greater than $\left(M_G\left(Q_{r,\ 0}^*\right)+1\right)\left(M_H\left(Q_{v,\ 0}^*\right)+1\right)$ we have

$$\frac{h_v\left(Q_{v,\ 0}^*+\int_0^{Q_{v,\beta}^*} M_H(w)dw\right)+A_v\lambda}{\left(M_H\left(Q_{v,\ 0}^*\right)+1\right)}\frac{(1-\beta_0)A_r\lambda}{M_G\left(Q_{r,\ 0}^*\right)+1} \\ > \frac{h_v\left(Q_{v,\ }^*+\int_0^{Q_{v,\beta}^*} M_H(w)dw\right)+A_v\lambda}{\left(M_H\left(Q_{v,\ }^*\right)+1\right)}\frac{(1-\beta)A_r\lambda}{M_G\left(Q_{r,\ }^*\right)+1}. \tag{B.22}$$

■